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Noncommutative vector bundles over fuzzy $\mathbb{CP}^N$ and their covariant derivatives

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Abstract: We generalise the construction of fuzzy $\mathbb{CP}^N$ in a manner that allows us to access all noncommutative equivariant complex vector bundles over this space. We give a simplified construction of polarization tensors on S^2 that generalizes to complex projective space, identify Laplacians and natural noncommutative covariant derivative operators that map between the modules that describe noncommuative sections. In the process we find a natural generalization of the Schwinger-Jordan construction to $su(n)$ and identify composite oscillators that obey a Heisenberg algebra on an appropriate Fock space.

Keywords: Discrete and Finite Symmetries, Solitons, Monopoles and Instantons, Matrix Models, Non-Commutative Geometry, Vector Bundles, Fuzzy.

Contents

1. Introduction

Noncommutative geometry [1 − 4] has become an active area of research in recent years providing potential new physics due to modifications of the underlying structure of space-time [5, 6]. Such geometries naturally arise in string theory [7]. A closely related development is that of fuzzy physics where finite matrix algebras are used to approximate the algebra of functions on a manifold. Action functionals built from these matrix algebras provide an alternative to lattice actions in the regularization of field theories and are especially natural for field theories on noncommutative spaces. Fuzzy spaces have also been pursued as potential spaces for the internal space in Kaluza-Klein reductions [8 − 10]. See [11] for a review of aspects of the fuzzy approach and [12, 13] for noncommutative field theory. The connection between matrix models and string theory is reviewed in [14] and the relation of fuzzy geometry to the quantum Hall effect is reviewed in [15].

As a regularization of quantum field theory, the fuzzy approach replaces the infinite number of degrees of freedom of the Euclidean quantum field with the finite number of degrees of freedom associated with a direct sum of modules over a finite simple matrix algebra, where the finite matrix algebra approximates the algebra of functions of the underlying compact Euclidean space. Of course, by Wedderburn's theorem all simple finite dimensional matrix algebras over the complex numbers are isomorphic and characterized only by the matrix dimension, so the geometry must be encoded directly in the action of the field theory. For scalar fields, it is the Laplacian that does this and hence, once one has specified the allowed matrix dimensions and the action for a free scalar field, the fuzzy, and hence limiting commutative, geometries are determined. In the case of spinor fields, the relevant operator is the Dirac operator which encodes the spin geometry [1] of the space.

A simple method of constructing a large variety of fuzzy spaces is as coadjoint orbits of compact Lie groups. In this way, the fuzzy space preserves all the isometries of the limiting commutative geometry even at the finite level and the commutative geometry is recovered in the limit of infinite size matrices.[1] This is a radical shift from the canonical lattice regularisation that is already a highly developed tool for quantum field theory. Its potential advantages are, however, sufficiently promising that it warrants attention in its own right. The principal advantages arise for models with Fermions, where the Dirac operators are significantly simpler than their lattice relatives and furthermore the fuzzy models avoid the difficulties of Fermion doubling [16]. Preliminary numerical studies of scalar field theories on fuzzy spaces have been performed in [17 − 19, 21, 20]. Fuzzy gauge fields on S^2_F have been discussed by several authors [22 − 24] and on $\mathbb{CP}^2$ in [25]. Recent numerical studies of such gauge theories can be found in [26] and [27].

The archetypical fuzzy space is the fuzzy sphere [28 − 30], though any manifold which can be generated as the coadjoint orbit of a compact Lie group should have a fuzzy description. See Arnlind *et al* [31] for a fuzzy torus construction not based on coadjoint orbits.[2] Explicit descriptions of fuzzy $\mathbb{CP}^N$, fuzzy unitary Grassmannians and fuzzy complex quadrics already exist [32 − 34]. There are also constructions of "fuzzy" spheres of dimension greater than two [35, 36] but all of these models involve additional degrees of freedom and require the introduction of some technique to decouple these modes.

To pin down the geometry, one must focus on the scalar Laplacian (or Dirac operator). In the literature there exist two distinct prescriptions for the Laplacian on the fuzzy sphere. The simplest prescription takes the Laplacian as the quadratic Casimir $\Delta = \hat{\mathcal{L}}_a\hat{\mathcal{L}}_a$ where $\hat{\mathcal{L}}_a = [\hat{L}_a, \cdot]$ and the $\hat{L}_a$ are the generators of $su(2)$ in the irreducible representation of dimension $L+1$ where L is any positive integer. This Laplacian has a natural generalization to $\mathbb{CP}^N$, where again the quadratic Casimir of $\mathrm{SU}(N+1)$ in the irreducible representation corresponding to the matrix size is used. However, in [37] an alternative prescription for the Laplacian is given in the case of S^2_F. Here the Laplacian is given as $\Delta = \frac{1}{2}(\hat{K}_-\hat{K}_+ + \hat{K}_+\hat{K}_-)$ in terms of operators (see (2.42) below) $\hat{K}_-$ and $\hat{K}_+$ which together with $\hat{K}_0$ satisfy the $su(2)$ algebra. In the commutative case differential operators corresponding to $\hat{\mathcal{L}}_a$ are the right invariant generators of SU(2) on itself, while those corresponding to $\hat{K}_+, \hat{K}_-$ and $\hat{K}_0$ are the corresponding generators of the left invariant vector fields. On coset spaces G/H the left and right invariant generators of G play very different roles: The right invariant generators act as Lie derivatives while the left invariant generators become covariant derivatives. In the fuzzy setting this is reflected in the fact that the operator images of right invariant generators, i.e. $\hat{\mathcal{L}}_a$, provide the generators of the adjoint action of the group G acting on the matrix algebra while the operators $\hat{K}_+$ and $\hat{K}_-$ change

[1] Due to the presence of all the continuous isometries the fuzzy theory is already continuum.

[2] In this torus example the relevant Laplacian has yet to be constructed.

the matrix sizes with $\hat{K}_+$ mapping from square $(L+1)\times(L+1)$ matrices to non-square $(L+2)\times L$ matrices and $\hat{K}_-$ mapping to $L\times(L+2)$ matrices. These latter matrices can be interpreted as projective modules over the fuzzy sphere and allow one to access line bundles over the sphere in a very natural way [37]. Also, it is straightforward to construct the Dirac operator and action functionals for spinors once these operators are known. The principal goal of this paper will be to give the construction of the corresponding operators for any complex projective space. We will see that this will involve us in an alternative construction of $\mathbb{CP}^N_F$, which opens up a variety of possibilities for the addition of structure to these spaces.

In section 2 we review the construction of both S^2_F, and of topologically nontrivial field configurations on this space and review the construction of the operators $\hat{K}_\pm$. Along the way we give a novel construction of polarization tensors for both square and non-square matrices (see [38] for the standard construction). In section 3 we repeat the construction for $\mathbb{CP}^N$ ending the section with a brief description of noncommutative line bundles over $\mathbb{CP}^N$. Section 4 introduces composite operators which surprisingly end up obeying the Heisenberg algebra on an appropriate reduced Fock space. The setting involves a natural generalization of the Schwinger-Jordan construction to $su(n)$. Section 5 gives the operators $\hat{K}_\imath$ and $\hat{K}_{\bar\imath}$ that generalize $\hat{K}_\pm$ and map between noncommutative vector bundles and describes the modules corresponding to these bundles. Section 6 contains our conclusions. Some technical results needed in the text are obtained in appendices.

2. The fuzzy sphere, S^2_F and its noncommutative line bundles

We begin by focusing on the fuzzy sphere $S^2_F\cong\mathbb{CP}^1_F$ using an approach that easily generalizes to other spaces. The generalization to $\mathbb{CP}^N$ will be pursued in subsequent sections.

Let a^α, $\alpha=1,2$ be a doublet of annihilation operators that annihilate the Fock vacuum $|0\rangle$ and let $a^\dagger_\beta$ (the Hermitian conjugate of a^β) be a conjugate pair of creation operators with the two doublets satisfying the Heisenberg commutation relations[3]

$$[a^\alpha,a^\beta]=[a^\dagger_\alpha,a^\dagger_\beta]=0\qquad\text{and}\qquad[a^\alpha,a^\dagger_\beta]=\delta^\alpha_\beta\ . \tag{2.1}$$

The Fock space $\mathcal{F}$ freely generated by the creation operators, $a^\dagger_\alpha$, is spanned by the orthonormal vectors

$$|n_1,n_2\rangle=\frac{1}{\sqrt{n_1!n_2!}}(a^\dagger_1)^{n_1}(a^\dagger_2)^{n_2}|0\rangle. \tag{2.2}$$

The Schwinger-Jordan construction then gives operators

$$\hat{N}=a^\dagger a\qquad\text{and}\qquad\hat{L}_a=a^\dagger\frac{\sigma_a}{2}a \tag{2.3}$$

which satisfy the $u(2)$ algebra

$$[\hat{L}_a,\hat{L}_b]=i\epsilon_{abc}\hat{L}_c,\qquad[\hat{L}_a,\hat{N}]=0. \tag{2.4}$$

[3]It will be convenient to use the metric $\delta_{\alpha\overline{\beta}}$ and its inverse $\delta^{\alpha\overline{\beta}}$ on $\mathbb{C}^2$ to raise and lower indices, so that $a^\alpha=\delta^{\alpha\overline{\beta}}a_{\overline{\beta}}$. Of course the distinction between upper and lower case indices for $su(2)$ is somewhat trivial as the representations are unitarily equivalent. However, for later use it will be convenient to maintain a consistent notation that extends to $su(N+1)$.

The raising and lowering operators $\hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2$ and $\hat{L}_0 = \hat{L}_3$ are explicitly

$$\hat{L}_+ = a_1^\dagger a^2, \qquad \hat{L}_- = a_2^\dagger a^1, \qquad \hat{L}_0 = \frac{1}{2}(a_1^\dagger a^1 - a_2^\dagger a^2), \tag{2.5}$$

and the algebra can equally be written

$$[\hat{L}_0, \hat{L}_\pm] = \pm L_\pm, \quad [\hat{L}_+, \hat{L}_-] = 2\hat{L}_0, \quad [\hat{N}, \hat{L}_\pm] = [\hat{N}, \hat{L}_0] = 0. \tag{2.6}$$

Since the $\hat{L}_a$ commute with $\hat{N}$ we can decompose $\mathcal{F}$ into a direct sum of eigenspaces of $\hat{N}$ as

$$\mathcal{F} = \bigoplus_{L=0}^{\infty} \mathcal{F}_L. \tag{2.7}$$

The subspace of states $\mathcal{F}_L$ of the Fock space $\mathcal{F}$ is the span of the $L+1$ vectors $|n_1, n_2\rangle$ with $n_1 + n_2 = L$, i.e.

$$\mathcal{F}_L = span\{|n_1, n_2\rangle \mid \quad n_1 + n_2 = L\}. \tag{2.8}$$

This space is a representation space of the unitary irreducible $L+1$ dimensional representation $(spin - \frac{L}{2})$ of $su(2)$ on which the number operator $\hat{N}$ and quadratic Casimir $\hat{L}_a\hat{L}_a$ take the numerical values L and $\frac{L}{2}(\frac{L}{2}+1)$, respectively.

Equally one can consider the dual Fock space $\mathcal{F}^*$ with vacuum vector $\langle 0|$ where $\langle 0|0\rangle = 1$ and the restricted dual subspaces $\mathcal{F}_L^*$. The algebra associated with the fuzzy geometry is realized as the linear span of $\mathcal{F}_L \otimes \mathcal{F}_L^*$; it is isomorphic to the algebra of $(L+1)$-dimensional matrices. The norm on the matrix algebra is taken to be the trace norm. Define $\hat{\mathcal{L}}_a = \hat{L}_a^{\mathrm{L}} - \hat{L}_a^{\mathrm{R}}$, with $\hat{L}_a^{\mathrm{L}}$ and $-\hat{L}_a^{\mathrm{R}}$ the $su(2)$ generators acting on $\mathcal{F}$ and $\mathcal{F}^*$ respectively. The geometry of the round fuzzy sphere is then fixed by specifying the Laplacian to be

$$\Delta = \hat{\mathcal{L}}_a\hat{\mathcal{L}}_a = \left(\hat{L}_a\hat{L}_a\right)^{\mathrm{L}} \otimes \mathbf{1} + \mathbf{1} \otimes \left(\hat{L}_a\hat{L}_a\right)^{\mathrm{R}} - 2\hat{L}_a^{\mathrm{L}} \otimes \hat{L}_a^{\mathrm{R}}. \tag{2.9}$$

Let us pause and examine this construction in more detail. Since, for our purposes, we will not need to diagonalize $\hat{L}_0$ it is more convenient to work in a basis where we do not distinguish between $a_1^\dagger$ and $a_2^\dagger$, but rather we will leave the SU(2) symmetry manifest. Our preferred basis for $\mathcal{F}_L$ is the set of vectors

$$|\boldsymbol{\alpha}\rangle = |\alpha_1, \cdots, \alpha_L\rangle = \frac{1}{\sqrt{L!}} a_{\alpha_1}^\dagger \cdots a_{\alpha_L}^\dagger |0\rangle\,, \tag{2.10}$$

which satisfy the orthogonality relation

$$\langle \boldsymbol{\beta}|\boldsymbol{\alpha}\rangle = \mathcal{S}_{\overline{\boldsymbol{\beta}}\boldsymbol{\alpha}} = \mathcal{S}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} \tag{2.11}$$

or more explicitly

$$\langle \beta_1, \cdots, \beta_L|\alpha_1, \cdots, \alpha_L\rangle = \mathcal{S}_{\alpha_1\cdots\alpha_L}^{\beta_1\ldots\beta_L} = \frac{1}{L!}\delta_{\{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_L\}}^{\beta_L} \tag{2.12}$$

where $\mathcal{S}_{\alpha_1\cdots\alpha_L}^{\beta_1\ldots\beta_L}$ is the projector onto totally symmetric tensors.

A basis for the finite dimensional algebra is provided by a pairing of vectors from $\mathcal{F}_L$ and $\mathcal{F}_L^*$ of the form $\{|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\beta}|\}$ and we denote the space spanned by this basis as $\mathcal{F} \otimes \mathcal{F}^*$

$$\mathcal{F} \otimes \mathcal{F}^* = span\{|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\beta}|\}. \tag{2.13}$$

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A general matrix is then given by

$$\mathbf{M} = \frac{1}{L!} M^{\alpha_1\cdots\alpha_L}{}_{\beta_1\cdots\beta_L} a^\dagger_{\alpha_1}\cdots a^\dagger_{\alpha_L}\,|0\rangle\,\langle 0|\,a^{\beta_1}\cdots a^{\beta_L} = M^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\beta}|\ . \tag{2.14}$$

The Laplacian is represented as in (2.9) where, in terms of raising and lowering operators, we have:

$$\hat{\mathcal{L}}_\pm \mathbf{M} = [\hat{L}_\pm, \mathbf{M}], \qquad \hat{\mathcal{L}}_0 \mathbf{M} = [\hat{L}_0, \mathbf{M}]. \tag{2.15}$$

In particular the matrix

$$\mathbf{1} = |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| = |\alpha_1,\cdots\alpha_L\rangle\langle\alpha_1,\cdots\alpha_L| \tag{2.16}$$

is easily seen to be in the kernel of $\hat{\mathcal{L}}_\pm$ and $\hat{\mathcal{L}}_0$ and represents the unit operator. Taking the trace, we have

$$\langle\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle := \langle\alpha_1,\cdots\alpha_L|\alpha_L,\cdots\alpha_1\rangle = L+1 \tag{2.17}$$

the dimension of the matrix algebra. It is further useful to introduce the notation

$$|\boldsymbol{\alpha}_l,\boldsymbol{\gamma}_{L-l}\rangle\langle\boldsymbol{\gamma}_{L-l},\boldsymbol{\beta}_l| = |\alpha_1,\cdots,\alpha_l,\gamma_{l+1},\cdots,\gamma_L\rangle\langle\gamma_L,\cdots,\gamma_{l+1},\beta_l,\cdots\beta_1| \tag{2.18}$$

for basis elements where the $L-l$ indices $\boldsymbol{\gamma}_{L-l}$ are contracted.

The different eigenspaces of the Laplacian provide the polarization tensors $\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}$ with $l = 0\ldots L$, where all remaining contractions between $\boldsymbol{\alpha}_l$ and $\boldsymbol{\beta}_l$ have been removed. These are easily constructed by Gram-Schmidt orthogonalization. The polarization tensors then satisfy

$$(\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l})^\dagger = \mathbf{Y}_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l} \quad \text{and} \quad \frac{Tr}{L+1}(\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}\mathbf{Y}_{\boldsymbol{\sigma}_{l'}}{}^{\boldsymbol{\tau}_{l'}}) = \delta_{ll'}\mathcal{P}_{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_{l'}}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_{l'}} \tag{2.19}$$

where $\mathcal{P}_{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_{l'}}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_{l'}}$ is the projector onto symmetric traceless tensors, i.e. it removes all traces between $\boldsymbol{\alpha}_l$ and $\boldsymbol{\beta}_l$ in (2.18). For example

$$\mathbf{Y} = \mathbf{1}, \qquad \mathbf{Y}_\alpha{}^\beta = \sqrt{\frac{6L}{L+2}}\left(|\alpha,\boldsymbol{\gamma}_{L-1}\rangle\langle\beta,\boldsymbol{\gamma}_{L-1}| - \frac{1}{2}\delta_\alpha{}^\beta\mathbf{1}\right) \tag{2.20}$$

and the polarization tensor, $\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}$, for angular momentum l has free indices $\boldsymbol{\alpha}_l$ and $\boldsymbol{\beta}_l$ with eigenvalue $l(l+1)$ for the Laplacian (2.9), i.e.

$$\hat{\mathcal{L}}^2\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l} = l(l+1)\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}\ . \tag{2.21}$$

The construction has a very clean group theoretical meaning: The decomposition into polarization tensors is the decomposition of the tensor product representation into irreducible representations and is expressed in Young diagrams as

$$\overbrace{\boxed{\ }\boxed{..}\boxed{\ }}^{L} \otimes \overbrace{\boxed{\ }\boxed{..}\boxed{\ }}^{L} = \mathbf{1} \oplus \boxed{\ }\boxed{\ } \oplus\cdots\oplus \overbrace{\boxed{\ }\boxed{..}\boxed{..}\boxed{\ }}^{2l} \oplus\cdots\oplus \overbrace{\boxed{\ }\boxed{..}\boxed{..}\boxed{\ }}^{2L}\ . \tag{2.22}$$

We see that increasing L by one adds one additional multiplet to (2.22). Our complete set of polarization tensors is

$$\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l} = \frac{\sqrt{L+1}}{\sqrt{Q(l,L)}}\mathcal{P}_{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_l}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_l}|\boldsymbol{\tau}_l,\boldsymbol{\gamma}_{L-l}\rangle\langle\boldsymbol{\sigma}_l,\boldsymbol{\gamma}_{L-l}|\ , \tag{2.23}$$

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where $\mathcal{P}_{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_l}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_l}$ is the projector onto the irreducible representation

$$\overbrace{\boxed{}\boxed{\cdot\cdot}\boxed{\cdot\cdot}\boxed{}}^{2l}\,, \tag{2.24}$$

i.e. it removes all traces associated with contractions of the free indices of $\mathcal{F}_L$ and $\mathcal{F}_L^*$. The coefficient $Q(l,L)$ in the normalization arises due to the contracted oscillators. With $l = L$ before the application of the projector $\mathcal{P}_{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_l}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_l}$ there are no contracted oscillators and the normalizations obtained from (2.19) is $\sqrt{L+1}$ so $Q(L,L) = 1$.

The more general normalization can be obtained by observing that the contraction over the $L-l$ indices $\boldsymbol{\gamma}_{L-l}$ corresponds to the repeated embedding of a matrix with angular momentum cutoff k into a matrix with cutoff $k+1$ by adding contracted oscillators, the operation being repeated from $k = l+1$ up to $k = L$. Thus for $\mathbf{M} \in \mathrm{Mat_{k+1}}$,

$$a^\dagger_{\gamma_{k+1}} \mathbf{M} a^{\gamma_{k+1}} \in \mathrm{Mat_{k+2}} \tag{2.25}$$

and we see from the above discussion that the polarization tensor content has not changed so an embedded matrix still has angular momentum only up to k; the top angular momentum $l = k+1$ is naturally absent. Now if we rewrite (2.9) we obtain the relations:

$$\begin{aligned} a^\dagger_\alpha a^\beta \mathbf{M} a^\dagger_\beta a^\alpha &= \left(\hat{N}(\hat{N}+1) - \hat{\mathcal{L}}^2\right)\mathbf{M} \qquad \text{and} \\ a^\alpha a^\dagger_\beta \mathbf{M} a^\beta a^\dagger_\alpha &= \left((\hat{N}+1)(\hat{N}+2) - \hat{\mathcal{L}}^2\right)\mathbf{M}\,, \end{aligned} \tag{2.26}$$

where we used $[\hat{N}, \mathbf{M}] = 0$. In the first expression in (2.26) the matrix $\mathbf{M}$ is reduced before being re-embedded thus projecting out the top multiplet with $l = k$ from the spectrum of the Laplacian $\hat{\mathcal{L}}^2$, while in the second the matrix $\mathbf{M}$ is embedded before being reduced. From these and the fact that only one multiplet is added by increasing the cutoff by one we can deduce that the eigenvalues of $\hat{\mathcal{L}}^2$ are $l(l+1)$. Also the factor $Q(l,L)$ in the normalization of polarization tensors is obtained from (2.19) and given by

$$Q(l,L) = \frac{(l!)^2}{(2l+1)!}\frac{(L-l)!(L+l+1)!}{(L!)^2}\,, \tag{2.27}$$

with $Q(L,L) = 1$.

It is now easy to relate this formulation to one in terms of functions. A given matrix $\mathbf{M} \in \mathrm{Mat_{L+1}}$ can be expanded in polarization tensors,

$$\mathbf{M} = \sum_{l=0}^{L} M^{\boldsymbol{\alpha}_l}{}_{\boldsymbol{\beta}_l} \mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}\;. \tag{2.28}$$

Define the *symmetric symbol density* as the matrix $\rho_L(\bar{z}, z)$

$$\rho_L(\bar{z}, z) = \sum_{l=0}^{L} Y_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}(\bar{z}, z)\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l} \tag{2.29}$$

where

$$Y_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}(\bar{z}, z) = \frac{\sqrt{(2l+1)!}}{l!}\mathcal{P}_{\boldsymbol{\beta}_l,\boldsymbol{\sigma}_l}{}^{\boldsymbol{\alpha}_l,\boldsymbol{\tau}_l}\,\bar{z}_{\tau_1}\dots\bar{z}_{\tau_l}z^{\sigma_1}\dots z^{\sigma_l} \tag{2.30}$$

are the ordinary spherical harmonics in a spinorial basis normalized such that

$$\frac{1}{\mathrm{Vol}(S^2)}\int_{S^2}\omega\;(Y_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}Y_{\boldsymbol{\tau}_{l'}}{}^{\boldsymbol{\sigma}_{l'}}) = \delta_{ll'}\mathcal{P}_{\boldsymbol{\beta}_l,\boldsymbol{\tau}_{l'}}{}^{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_{l'}}\,, \tag{2.31}$$

with ω the volume form and $\bar{z}_\alpha z^\alpha = 1$.

Then the trace

$$M(\bar{z}, z) = \frac{Tr}{L+1}\left(\rho_L(\bar{z}, z)\mathbf{M}\right) \tag{2.32}$$

gives a function on S^2 whose expansion in spherical harmonics has the same coefficients as the coefficients of the matrix in terms of the polarization tensors. We can approximate a function $f(\bar{z}, z) \in C^\infty(S^2)$ by the matrix

$$\mathbf{M}_f = \frac{1}{\mathrm{Vol}(S^2)}\int_{S^2}\omega\rho_L f\,, \tag{2.33}$$

which is a matrix whose coefficients in an expansion in polarization tensors coincides with f up to angular momentum L and all higher coefficients are projected to zero. If one substitutes the function $M(\bar{z}, z)$ obtained from $\mathbf{M}$ in (2.32) into (2.33) one recovers the matrix $\mathbf{M}$.

An equally good map to functions is provided by[4] the diagonal coherent state map [39]

$$M_L(\bar{z}, z) = \langle z, L|\mathbf{M}|L, z\rangle\,, \tag{2.34}$$

where we have taken the trace of

$$\begin{aligned} |z, L\rangle\langle L, z| &:= \frac{1}{L!}(z^\alpha a^\dagger_\alpha)^L|0\rangle\langle 0|(\bar{z}_\beta a^\beta)^L \\ &= \sum_{l=0}^{L}\frac{T_L^{1/2}(l)}{L+1}Y_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}(\bar{z}, z)\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}\,, \end{aligned} \tag{2.35}$$

which effects the simple replacement $a^\alpha \to z^\alpha$, $a^\dagger_\alpha \to \bar{z}_\alpha$ in (2.14) and the removal of $1/L!$.

The principal difference in the two maps to functions (2.29) and (2.35) is the presence in the latter of the coefficients

$$T_L(l) = \frac{L!(L+1)!}{(L-l)!(L+l+1)!} = \left(1 - \frac{l(l+1)}{L(L+1)}\right)^{-1}T_{L-1}(l)\,, \tag{2.36}$$

which alter the coefficients in the expansion of a function from those in the expansion of the corresponding matrix. Note that for $L \to \infty$ with l fixed $T_L(l) \to 1$ so the distortion disappears as $L \to \infty$. The image functions M_L and M are related by

$$M_L(\bar{z}, z) = \mathcal{T}_L^{1/2}(\mathcal{L}^2)M(\bar{z}, z)\,, \tag{2.37}$$

[4]The conventions here are set by noting that for $L = 1$ we have $D^{\frac{1}{2}}_{\alpha,\frac{1}{2}}(z, \bar{z}) = z^\alpha$ and $|z, 1\rangle = z^\alpha|\alpha\rangle = z^\alpha a^\dagger_\alpha|0\rangle = (\bar{z}_\alpha a^\alpha)^\dagger|0\rangle$, where $D^j_{m_1,m_2}(g)$ are the Wigner D-matrices for SU(2), and on the entire Fock space, $\mathcal{F}$, coherent states are defined as eigenvectors of the annihilation operator.

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where the rotationally invariant operator $\mathcal{T}_L(\mathcal{L}^2)$ has eigenvalues $T_L(l)$ [28, 40, 41]. The principal advantage of the coherent state map is the simplicity of the associated *-product [32, 42]. In contrast that induced by (2.29) is more complicated but its leading term in a large L expansion is the Poisson bracket, i.e. the symmetric part of the *-product vanishes in the leading term.

There is an alternative set of operators $\tilde{\mathcal{L}}_\pm$ to those in (2.15) obtained by interchanging both the roles of left and right and $+$ and $-$ to obtain new operators:

$$\tilde{\mathcal{L}}_\pm \mathbf{M} = -[\hat{L}_\mp, \mathbf{M}], \qquad \tilde{\mathcal{L}}_0 \mathbf{M} = -[\hat{L}_0, \mathbf{M}] \ . \tag{2.38}$$

These generators can be induced naturally by observing that the set of states is unchanged by transforming to the oscillators

$$a_\alpha := a^\beta \epsilon_{\beta\alpha} \equiv a^{\overline{\alpha}} , \tag{2.39}$$

i.e. substituting $a_1 \to -a^2$ and $a_2 \to a^1$. The set $a^\dagger_{\overline{\alpha}}$ generate precisely the same matrix algebra and with this substitution, we have

$$\begin{aligned} \tilde{L}_+ &= (a_1)^\dagger a_2 = -a_2^\dagger a^1 = -\hat{L}_- , \\ \tilde{L}_- &= (a_2)^\dagger a_1 = -a_1^\dagger a^2 = -\hat{L}_+ \\ \tilde{L}_0 &= \frac{1}{2}((a_1)^\dagger a_1 - (a_2)^\dagger a_2) = \frac{1}{2}(a_2^\dagger a^2 - a_1^\dagger a^1) = -\hat{L}_0 \end{aligned} \tag{2.40}$$

and $\tilde{\mathcal{L}}^2 = \hat{\mathcal{L}}^2$ so the resulting Laplacian is however unchanged. This reflects the fact complex conjugating a representation gives a unitarily equivalent one for $su(2)$.

There is yet a further realization of the Laplacian. This was first given in [37] in terms of the operators:[5]

$$\begin{aligned} \hat{K}_+ &:= (a^\dagger_\alpha)^{\mathrm{L}} ((a^\dagger)^\alpha)^{\mathrm{R}} &&: \mathcal{F}_L \otimes \mathcal{F}_L^* \longmapsto \mathcal{F}_{L+1} \otimes \mathcal{F}_{L-1}^* \\ \hat{K}_- &:= (a^\alpha)^{\mathrm{L}} (a_\alpha)^{\mathrm{R}} &&: \mathcal{F}_L \otimes \mathcal{F}_L^* \longmapsto \mathcal{F}_{L-1} \otimes \mathcal{F}_{L+1}^* \\ \hat{K}_0 &:= \tfrac{1}{2}(\hat{N}^{\mathrm{L}} - \hat{N}^{\mathrm{R}}) &&: \mathcal{F}_L \otimes \mathcal{F}_L^* \longmapsto 0 \end{aligned} \tag{2.41}$$

where the L and R superscripts indicate that the operators act on the left or right, i.e. on $\mathcal{F}_L$ or $\mathcal{F}_L^*$, respectively.

Note: These operators do not require that the left and right truncated Fock spaces have the same dimension, more generally, they act on bimodules of the form $\mathcal{F}_{n_{\mathrm{L}}} \otimes \mathcal{F}^*_{n_{\mathrm{R}}}$ which represent non-square matrices which are left modules for the algebra $\mathrm{Mat}_{\mathrm{n_L}+1}$ and right modules for $\mathrm{Mat}_{\mathrm{n_R}+1}$. For our purposes, it is convenient to denote such a generic element of a left module of $\mathrm{Mat}_{\mathrm{L}+1}$ by $\mathbf{M}_q$, where $\mathbf{M}_q \in \mathcal{F}_L \otimes \mathcal{F}^*_{L-q}$ and we have

$$\begin{aligned} \hat{K}_+ \mathbf{M}_q &= \epsilon^{\alpha\beta} a^\dagger_\beta \mathbf{M}_q a^\dagger_\alpha , \\ \hat{K}_- \mathbf{M}_q &= \epsilon_{\alpha\beta} a^\beta \mathbf{M}_q a^\alpha , \\ \hat{K}_0 \mathbf{M}_q &= \frac{1}{2}[\hat{N}, \mathbf{M}_q] = \frac{q}{2} \mathbf{M}_q . \end{aligned} \tag{2.42}$$

[5] One can multiply both $\hat{K}_+$ and $\hat{K}_-$ by opposite phases to get equivalent operators.

We see that $\hat{K}_+$ and $\hat{K}_-$ change the module structure as in (2.41) while $\hat{K}_0$ measures the non-squareness of a given bimodule and in particular, the matrix algebra can be identified with the kernel of $\hat{K}_0$ (i.e. with $q = 0$, $\mathbf{M} \in \mathrm{Mat}_{\mathrm{L}+1}$ is in the kernel of $\hat{K}_0$). Repeated applications of $\hat{K}_\pm$ map us further along the sequence of modules.

Although $\hat{K}_+$ and $\hat{K}_-$ take us out of the algebra of square matrices the products $\hat{K}_+\hat{K}_-$ and $\hat{K}_-\hat{K}_+$ do not. In general, one obtains

$$\begin{aligned} \hat{K}_+\hat{K}_-\mathbf{M}_q &= \hat{N}\mathbf{M}_q(\hat{N}+1) - a^\dagger_\alpha a^\beta \mathbf{M}_q a^\dagger_\beta a^\alpha \\ \hat{K}_-\hat{K}_+\mathbf{M}_q &= (\hat{N}+1)\mathbf{M}_q\hat{N} - a^\dagger_\alpha a^\beta \mathbf{M}_q a^\dagger_\beta a^\alpha \end{aligned} \tag{2.43}$$

and the operators $\hat{K}_\pm$ and $\hat{K}_0$ are easily seen to satisfy the $su(2)$ commutation relations. Similar manipulations for the Laplacian (2.9) acting on $\mathbf{M}_q$ yield

$$\hat{\mathcal{L}}^2\mathbf{M}_q = \left(L - \frac{q}{2}\right)\left(L - \frac{q}{2} + 1\right)\mathbf{M}_q - a^\dagger_\alpha a^\beta \mathbf{M}_q a^\dagger_\beta a^\alpha. \tag{2.44}$$

Furthermore, we have

$$\begin{aligned} \hat{K}^2\mathbf{M}_q &= \frac{1}{2}\left(\hat{K}_-\hat{K}_+ + \hat{K}_+\hat{K}_- + 2\hat{K}_0^2\right)\mathbf{M}_q \\ &= \left(L - \frac{q}{2}\right)\left(L - \frac{q}{2} + 1\right)\mathbf{M}_q - a^\dagger_\alpha a^\beta \mathbf{M}_q a^\dagger_\beta a^\alpha. \end{aligned} \tag{2.45}$$

Hence the potential Laplacians $\hat{\mathcal{L}}^2$, $\tilde{\mathcal{L}}^2$ and $\hat{K}^2$ are all equal and we are left with a unique option for the round Laplacian $\Delta = \hat{\mathcal{L}}^2 = \hat{K}^2$ on S^2_F given by (2.44).

As pointed out in [37] these latter non-square matrices capture topologically nontrivial field configurations on the fuzzy sphere and can be taken to be the noncommutative versions of holomorphic line bundles, with the eigenvalue of $\hat{K}_0$ given by $q/2$ and q counting the winding number, so that $q > 0$ describe $\overline{\mathcal{O}}(q)$ bundles and $q < 0$ describe $\mathcal{O}(-q)$ bundles. The Laplacian for these line bundles is naturally given by (2.45) while that based on the construction given in [43 − 46] is more cumbersome.

The generalization of (2.28) to non-square matrices[6] $\mathbf{M}_q$ is given by

$$\mathbf{M}_q = \sum_{l=0}^{L-q} M^{\boldsymbol{\alpha}_{l+q}}{}_{\boldsymbol{\beta}_l}\mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l} \tag{2.46}$$

and $\mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l}$ are the polarization tensors of $spin-(l+\frac{q}{2})$ which are obtained by projection onto the relevant representation in the decomposition of $\mathcal{F}_L \otimes \mathcal{F}^*_{n_\mathrm{R}}$ where $n_\mathrm{R} = L - q$, i.e.

$$\mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l} = \frac{\sqrt{n_\mathrm{R}+1}}{\sqrt{Q(l,q,L)}}\mathcal{P}_{\boldsymbol{\alpha}_{l+q},\boldsymbol{\sigma}_l}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_{l+q}}|\boldsymbol{\tau}_{l+q},\boldsymbol{\gamma}_{n_\mathrm{R}-l}\rangle\langle\boldsymbol{\sigma}_l,\boldsymbol{\gamma}_{n_\mathrm{R}-l}| \tag{2.47}$$

as in (2.23) above and normalized such that

$$\frac{1}{n_\mathrm{R}+1}Tr\left((\mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l})^\dagger\mathbf{D}_{\boldsymbol{\sigma}_{l'+q'}}{}^{\boldsymbol{\tau}_{l'}}\right) = \delta_{ll'}\delta_{qq'}\mathcal{P}_{\boldsymbol{\beta}_l,\boldsymbol{\sigma}_{l+q}}{}^{\boldsymbol{\alpha}_{l+q},\boldsymbol{\tau}_l} \tag{2.48}$$

[6]Polarization tensors for non-square matrices in the standard basis were constructed by Prešnajder in [38] and are readily related to those we present.

and

$$Q(l,q,L) = \frac{l!(l+q)!}{(2l+1+q)!}\frac{(L-q-l)!(L+l+1)!}{(L-q)!L!}. \tag{2.49}$$

The Laplacian Δ is diagonal on the polarization tensors and we have

$$\Delta \mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l} = \left(l+\frac{q}{2}\right)\left(l+\frac{q}{2}+1\right)\mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l}\,, \quad \text{with} \quad l=0,\dots L-q\,. \tag{2.50}$$

The *symbol density* analogous to (2.29) which now provides a map to equivariant sections of line bundles over S^2 is

$$\rho_{L,q}(\bar z, z) = \sum_{l=0}^{L-q} D_{\boldsymbol{\beta}_{l+q}}{}^{\boldsymbol{\alpha}_l}(\bar z, z)(\mathbf{D}_{\boldsymbol{\beta}_{l+q}}{}^{\boldsymbol{\alpha}_l})^\dagger = \sum_{l=0}^{L-q} D_{\boldsymbol{\beta}_{l+q}}{}^{\boldsymbol{\alpha}_l}(\bar z, z)\mathbf{D}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_{l+q}}\,, \tag{2.51}$$

where

$$D_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l}(\bar z, z) = \sqrt{\frac{(2l+1+q)!}{l!(l+q)!}}\mathcal{P}_{\boldsymbol{\alpha}_{l+q},\boldsymbol{\sigma}_l}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_{l+q}}\bar z_{\tau_1}\dots\bar z_{\tau_{l+q}}z^{\sigma_1}\dots z^{\sigma_l} \equiv \overline{D}^j_{m,\frac{q}{2}}(z,\bar z) \tag{2.52}$$

with $j = l+\frac{q}{2}$ and $D^j_{m,s}(z,\bar z)$ are the Wigner D-matrices.

The relevant coherent state map for matrices $\mathbf{M}_q$ is provided by

$$\begin{aligned} |z, n_\mathrm{R}\rangle\langle n_\mathrm{L}, z| &:= \frac{1}{\sqrt{n_\mathrm{R}!n_\mathrm{L}!}}(z^\alpha a^\dagger_\alpha)^{n_\mathrm{R}}|0\rangle\langle 0|(\bar z_\alpha a^\alpha)^{n_\mathrm{L}} \\ &= \sum_{l=0}^{n_\mathrm{R}} \frac{T_L^{1/2}(l,q)}{n_\mathrm{R}+1} D_{\boldsymbol{\beta}_{l+q}}{}^{\boldsymbol{\alpha}_l}(\bar z, z)(\mathbf{D}_{\boldsymbol{\beta}_{l+q}}{}^{\boldsymbol{\alpha}_l})^\dagger \end{aligned} \tag{2.53}$$

where $n_\mathrm{L} = L$ and $n_\mathrm{R} = L-q$ and

$$T_L(l,q) = \frac{L!(L-q+1)!}{(L-q-l)!(L+l+1)!}\,. \tag{2.54}$$

If we use the diagonal coherent state map (2.53) it is easy to establish the correspondence

$$\begin{aligned} &\left(\frac{1}{\sqrt{\hat N}}a^\dagger_\alpha\right)^\mathrm{L} \longmapsto \bar z_\alpha &\qquad &\left(a^\alpha\sqrt{\hat N}\right)^\mathrm{L} \longmapsto \frac{\partial}{\partial \bar z_\alpha} \\ &\left(a^\alpha\frac{1}{\sqrt{\hat N}}\right)^\mathrm{R} \longmapsto z^\alpha &\qquad &\left(\sqrt{\hat N}a^\dagger_\alpha\right)^\mathrm{R} \longmapsto \frac{\partial}{\partial z^\alpha}\,. \end{aligned}$$

The presence of $\sqrt{\hat N}$ in these expressions takes care of the normalization of the states.

Now that we have mapped matrices to functions and non-square matrices to equivariant sections of monopole bundles we are in a position to map the various differential operators and the Laplacian to their commutative analogues. However, for our later discussion it is useful to pause and make a small digression back to the commutative setting and review the relevant operators there.

Using the identification induced by (2.55) and the operators identified in appendix A we have

$$\hat{L}_+^{\rm L} = (a_1^\dagger a^2)^{\rm L} \longmapsto \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} = -\bar{L}_- \,, \quad \hat{L}_+^{\rm R} = (a_1^\dagger a^2)^{\rm R} \longmapsto z^2 \frac{\partial}{\partial z^1} = -L_+ \,, \tag{2.55}$$

$$\hat{L}_-^{\rm L} = (a_2^\dagger a^1)^{\rm L} \longmapsto \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} = -\bar{L}_+ \,, \quad \hat{L}_-^{\rm R} = (a_2^\dagger a^1)^{\rm R} \longmapsto z^1 \frac{\partial}{\partial z^2} = -L_- \,, \tag{2.56}$$

$$\hat{L}_0^{\rm L} = \frac{1}{2}\left((a_1^\dagger a^1)^{\rm L} - (a_2^\dagger a^2)^{\rm L}\right) \longmapsto \frac{1}{2}\left(\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}\right) = -\bar{L}_0 \,, \tag{2.57}$$

$$\hat{L}_0^{\rm R} = \frac{1}{2}\left((a_1^\dagger a^1)^{\rm R} - (a_2^\dagger a^2)^{\rm R}\right) \longmapsto \frac{1}{2}\left(z^1 \frac{\partial}{\partial z^1} - z^2 \frac{\partial}{\partial z^2}\right) = -L_0 \;. \tag{2.58}$$

so the operators $\hat{\mathcal{L}}_a$ map to the right invariant vector fields $\mathcal{L}_a$ as:

$$\hat{\mathcal{L}}_+ \longmapsto \mathcal{L}_+ = L_+ - \bar{L}_- \,, \qquad \hat{\mathcal{L}}_- \longmapsto \mathcal{L}_- = L_- - \bar{L}_+ \,, \qquad \hat{\mathcal{L}}_0 \longmapsto \mathcal{L}_0 = L_0 - \bar{L}_0 \,. \tag{2.59}$$

Similarly using the coherent state map (2.53) and eq. (A.10) we find

$$\hat{K}_+ \longmapsto -\sqrt{\frac{L+1}{L-q}}\mathcal{R}_- \,, \qquad \hat{K}_- \longmapsto -\sqrt{\frac{L-q+1}{L}}\mathcal{R}_+ \,, \qquad \hat{K}_0 \longmapsto -\mathcal{R}_0 \,, \tag{2.60}$$

where the normalizations on the right do not affect the $su(2)$ commutation relations. These factors are harmless and could be removed by replacing

$$\hat{K}_+ \longmapsto \frac{1}{\sqrt{\hat{N}^{\rm L}}}\hat{K}_+\sqrt{\hat{N}^{\rm R}} \quad \text{and} \quad \hat{K}_- \longmapsto \frac{1}{\sqrt{\hat{N}^{\rm R}}}\hat{K}_-\sqrt{\hat{N}^{\rm L}} \;. \tag{2.61}$$

and $\hat{K}_0$ is unaffected. Finally with the above, we have of course $\Delta \longmapsto -\nabla^2$.

We could alternatively have built our map to functions using $a^\dagger_{\bar{\alpha}}$, i.e. via

$$|\,\bar{z}, n_{\rm R}\,\rangle\,\langle\, z, n_{\rm L}\,| = \frac{1}{\sqrt{n_{\rm R}!n_{\rm L}!}}(\bar{z}_\alpha a^\dagger_{\bar{\alpha}})^{n_{\rm R}}\,|0\rangle\,\langle 0|\,(z^\alpha a_\alpha)^{n_{\rm L}} \;. \tag{2.62}$$

This would end up in replacing z by $\bar{z}$, or equivalently replacing $\mathbf{M}$ with $\mathbf{M}^\dagger$ in the map between matrices and functions.

A principal result of this paper is the generalization of the operators $\hat{K}_\pm$ to the case of $\mathbb{CP}^N$ and we will see that it will give us access to projective modules that correspond to the algebra of functions tensored with holomorphic line bundles and complex vector bundles over $\mathbb{CP}^N$.

3. Fock space construction of $\mathbb{CP}^N_F$

The above discussion is very easily adapted to $\mathbb{CP}^N$ by considering a^α, a multiplet of $N+1$ oscillators. In this case the construction analogous to (2.3) and (2.4) gives the $u(N+1)$ algebra based on the operators

$$\hat{N} = a^\dagger a, \qquad \text{and} \qquad \hat{L}_a = a^\dagger \frac{\lambda_a}{2} a \tag{3.1}$$

where λ_a are the Gell-Mann matrices[7] of SU$(N+1)$ with a running from 1 to $(N+1)^2-1$. We can again decompose the Fock space, $\mathcal{F}$, generated freely by $a^\dagger_\alpha$, into a direct sum of finite dimensional spaces $\mathcal{F}_L$ carrying the irreducible representation corresponding to L-fold symmetric tensor product of the fundamental of $su(N+1)$, which we labeled by the eigenvalue of $\hat{N}$.

We similarly define states spanning $\mathcal{F}_L$ to be

$$|\alpha_1,\cdots\alpha_L\rangle = \frac{1}{\sqrt{L!}} a^\dagger_{\alpha_1}\cdots a^\dagger_{\alpha_L}|0\rangle. \tag{3.3}$$

The unit matrix is represented by

$$\mathbf{1} = |\alpha_1,\cdots\alpha_L\rangle\langle\alpha_1,\cdots\alpha_L| \tag{3.4}$$

and has trace

$$\langle\alpha_1,\cdots\alpha_L|\alpha_1,\cdots\alpha_L\rangle = d_N(L) = \frac{(N+L)!}{N!L!}\,, \tag{3.5}$$

where $d_N(L)$ is the dimension of the L-fold symmetric tensor product representation of SU$(N+1)$. A generic matrix $\mathbf{M}$ is as in the case of S^2_F given by

$$\mathbf{M} = \frac{1}{L!} M^{\alpha_1\cdots\alpha_L}{}_{\beta_1\cdots\beta_L} a^\dagger_{\alpha_1}\cdots a^\dagger_{\alpha_L}\,|\,0\,\rangle\,\langle\,0\,|\,a^{\beta_1}\cdots a^{\beta_L}. \tag{3.6}$$

The geometry of $\mathbb{CP}^N_F$ can then be specified by building the derivatives $\hat{\mathcal{L}}_a$, as we did for S^2_F, from the commutator action

$$\hat{\mathcal{L}}_a\mathbf{M} = [\hat{L}_a,\mathbf{M}], \qquad \text{where} \qquad \hat{L}_a = a^\dagger\frac{\lambda_a}{2}a \tag{3.7}$$

with a running from 1 to $(N+1)^2-1$. The Laplacian given by the quadratic Casimir $\Delta = \hat{\mathcal{L}}_a\hat{\mathcal{L}}_a$, can be expressed in the form (2.44) as

$$\hat{\mathcal{L}}^2\mathbf{M} = \Big(L(L+N)\mathbf{M} - a^\dagger_\alpha a^\beta\mathbf{M}a^\dagger_\beta a^\alpha\Big)\ . \tag{3.8}$$

This is easily obtained using

$$(\lambda_a)_{\overline{\alpha}\beta}(\lambda_a)_{\overline{\mu}\nu} = 2\delta_{\overline{\alpha}\nu}\delta_{\overline{\mu}\beta} - \frac{2}{N+1}\delta_{\overline{\alpha}\beta}\delta_{\overline{\mu}\nu}\,, \tag{3.9}$$

yielding

$$2\hat{L}^{\rm L}_a\otimes\hat{L}^{\rm R}_a = \left(a^\dagger_\alpha a^\beta\right)^{\rm L}\otimes\left(a^\dagger_\beta a^\alpha\right)^{\rm R} - \frac{\hat{N}^{\rm L}\otimes\hat{N}^{\rm R}}{N+1} \tag{3.10}$$

and

$$\hat{C}_2 = \hat{L}^{\rm L}_a\hat{L}^{\rm L}_a = \frac{N}{2(N+1)}\hat{N}(\hat{N}+N+1). \tag{3.11}$$

[7] In our conventions the Gell-Mann matrices in the anti-fundamental are denoted $\overline{\lambda}_a$ and given by

$$(\overline{\lambda}_a)_{\alpha\overline{\beta}} = -(\lambda_a)_{\overline{\beta}\alpha}. \tag{3.2}$$

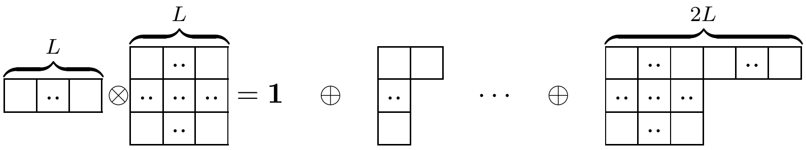
More generally, we can write

$$\hat{\mathcal{L}}^2 = \hat{C}_2 \otimes \mathbf{1} + \mathbf{1} \otimes \hat{C}_2 + \frac{\hat{N} \otimes \hat{N}}{N+1} - (a^\dagger_\alpha a^\beta) \otimes (a^\dagger_\beta a^\alpha)\ . \tag{3.12}$$

From this we see the analogues of (2.26) become

$$\begin{aligned} a^\dagger_\alpha a^\beta \mathbf{M} a^\dagger_\beta a^\alpha &= \Big(L(L+N) - \hat{\mathcal{L}}^2\Big)\,\mathbf{M} \\ a^\alpha a^\dagger_\beta \mathbf{M} a^\beta a^\dagger_\alpha &= \Big((L+1)(L+1+N) - \hat{\mathcal{L}}^2\Big)\,\mathbf{M}\ . \end{aligned} \tag{3.13}$$

As in the case of S^2_F the polarization tensors are given by the decomposition of the tensor product $\mathcal{F}_L \otimes \mathcal{F}^*_L$, but now into irreducible representations of $su(N+1)$. Since $\mathcal{F}_L$ and $\mathcal{F}^*_L$ carry the L-fold symmetric tensor product representations of the fundamental and anti-fundamental of $su(N+1)$, respectively, the relevant group theory decomposition is



(3.14)

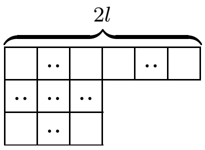
and the decomposition into polarization tensors is a realization of this decomposition where the polarization tensors with $2l$-free indices denoted $\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}$. In analogy with (2.23), the $\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}$ for fuzzy $\mathbb{CP}^N$ are given by

$$\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l} = \frac{\sqrt{d_N(L)}}{\sqrt{Q_N(l,L)}} \mathcal{P}_{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_l}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_l} |\boldsymbol{\tau}_l, \boldsymbol{\gamma}_{L-l}\rangle\langle\boldsymbol{\sigma}_l, \boldsymbol{\gamma}_{L-l}| \tag{3.15}$$

where

$$Q_N(l,L) = \left(\frac{l!}{L!}\right)^2 \frac{(L-l)!(L+l+N)!}{(2l+N)!} \tag{3.16}$$

and $\mathcal{P}_{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_l}{}^{\boldsymbol{\beta}_l,\boldsymbol{\tau}_l}$ is the projector onto the representation



. (3.17)

The projector removes all traces corresponding to the lower dimensional representations in (3.14).

Explicitly, $\mathbf{Y} = \mathbf{1}$ for $l = 0$ and for $l = 1$ we have:

$$\mathbf{Y}_\alpha{}^\beta = \sqrt{\frac{(N+1)(N+2)L}{L+N+1}} \left(|\boldsymbol{\alpha}, \boldsymbol{\gamma}_{L-1}\rangle\langle\boldsymbol{\beta}, \boldsymbol{\gamma}_{L-1}| - \frac{1}{N+1}\delta_\alpha{}^\beta \mathbf{1}\right)\ . \tag{3.18}$$

The decomposition of a matrix $\mathbf{M} \in \mathrm{Mat}_{\mathrm{d_N(L)}}$ in terms of polarization tensors is as before

$$\mathbf{M} = \sum_{l=0}^{L} M^{\boldsymbol{\alpha}_l}{}_{\boldsymbol{\beta}_l} \mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l} \tag{3.19}$$

and the map to functions given by

$$M(\bar{z},z) = \frac{Tr}{d_N(L)}(\rho_L(\bar{z},z)\mathbf{M}) \tag{3.20}$$

with the symmetric symbol density, $\rho_L(\bar{z},z)$, given by

$$\rho_L(\bar{z},z) = \sum_{l=0}^{L} Y_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}(\bar{z},z)\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l} \tag{3.21}$$

and the normalizations are the generalization to $\mathbb{CP}^N$ of (2.19) and (2.31), i.e.

$$\frac{Tr}{d_N(L)}(\mathbf{Y}_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}\mathbf{Y}_{\boldsymbol{\tau}_{l'}}{}^{\boldsymbol{\sigma}_{l'}}) = \delta_{ll'}\mathcal{P}_{\boldsymbol{\beta}_l,\boldsymbol{\tau}_{l'}}{}^{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_{l'}} \tag{3.22}$$

and

$$\frac{1}{\mathrm{Vol}(\mathbb{CP}^N))}\int_{\mathbb{CP}^N}\omega^N(Y_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}Y_{\boldsymbol{\tau}_{l'}}{}^{\boldsymbol{\sigma}_{l'}}) = \delta_{ll'}\mathcal{P}_{\boldsymbol{\beta}_l,\boldsymbol{\tau}_{l'}}{}^{\boldsymbol{\alpha}_l,\boldsymbol{\sigma}_{l'}}\ , \tag{3.23}$$

where ω^N is the volume form on $\mathbb{CP}^N$ and

$$Y_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}(\bar{z},z) = \frac{\sqrt{(2l+N)!}}{l!\sqrt{N!}}\mathcal{P}_{\boldsymbol{\beta}_l,\boldsymbol{\sigma}_l}{}^{\boldsymbol{\alpha}_l,\boldsymbol{\tau}_l}\bar{z}_{\tau_1}\dots\bar{z}_{\tau_l}z^{\sigma_1}\dots z^{\sigma_l}\ . \tag{3.24}$$

Conversely, the function $f(\bar{z},z)\in C^\infty(\mathbb{CP}^N)$ is approximated by the matrix

$$\mathbf{M}_f = \frac{1}{\mathrm{Vol}(\mathbb{CP}^N)}\int_{\mathbb{CP}^N}\omega^N\rho_L f\ . \tag{3.25}$$

Mapping $\mathbf{M}_f$ in (3.25) back to functions using (3.19) will result in an approximation to the function f where the coefficients of all representations of $\mathrm{SU}(N+1)$, in f, that lie above the cutoff representation of dimension $d_N(L)$ are projected to zero.

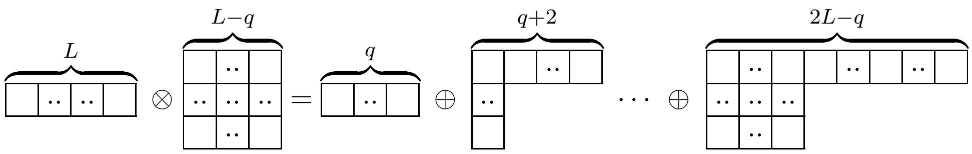
Again the corresponding coherent state is provided by

$$\begin{aligned}|z,L\rangle\langle L,z| &= \sum_{l=0}^{L}\frac{T_L^{1/2}(l,N)}{d_N(L)}Y_{\boldsymbol{\beta}_l}{}^{\boldsymbol{\alpha}_l}(\bar{z},z)\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l} \\ &= \frac{1}{L!}(z^\alpha a_\alpha^\dagger)^L|0\rangle\langle 0|(\bar{z}_\alpha a^\alpha)^L\ .\end{aligned} \tag{3.26}$$

With (3.13), the eigenvalues of the $\mathrm{SU}(N+1)$ invariant operator $\mathcal{T}_L(\mathcal{L}^2,N)$ generalizing (2.37) to the case of $\mathbb{CP}^N$ are found to be

$$T_L(l,N) = \frac{L!(L+N)!}{(L-l)!(L+l+N)!}\ . \tag{3.27}$$

More generally, for non-square matrices $\mathcal{F}_L\otimes\mathcal{F}^*_{L-q}$ the relevant group theory decomposition is given by



(3.28)

which in terms of Dynkin indices reads

$$(L, \ldots, 0, 0) \otimes (0, 0, \ldots, L-q) = \bigoplus_{l=0}^{L-q} (l+q, 0, \ldots, 0, l) \ . \tag{3.29}$$

Observe that increasing the cutoff L by one adds just one additional representation. Then using (3.12) we have, for the $\mathbf{M}_q \in \mathcal{F}_L \otimes \mathcal{F}^*_{L-q}$,

$$\begin{aligned} a^\dagger_\alpha a^\beta \mathbf{M}_q a^\dagger_\beta a^\alpha &= \left(C_2(N+1, L, L-q) - \hat{\mathcal{L}}^2\right) \mathbf{M}_q \\ a^\alpha a^\dagger_\beta \mathbf{M}_q a^\beta a^\dagger_\alpha &= \left(C_2(N+1, L+1, L+1-q) - \hat{\mathcal{L}}^2\right) \mathbf{M}_q \, , \end{aligned} \tag{3.30}$$

where

$$C_2(N+1, n_\mathrm{L}, n_\mathrm{R}) = \frac{1}{2}\left(n_\mathrm{L}(n_\mathrm{R}+N) + n_\mathrm{R}(n_\mathrm{L}+N)\right) + \frac{N}{2(N+1)}(n_\mathrm{L}-n_\mathrm{R})^2 \ . \tag{3.31}$$

The operator $a^{\beta \mathrm{L}} a^{\dagger\,\mathrm{R}}_\beta$ involves the reduction of the cutoff by one and so projects out the top representation in (3.28). The matrix is subsequently re-embedded using $a^{\dagger\,\mathrm{L}}_\alpha \otimes a^{\alpha\mathrm{R}}$. Hence $C_2(N+1, n_\mathrm{L}, n_\mathrm{R})$ is the eigenvalue of the quadratic Casimir of $su(N+1)$ on the representation $(L, 0, \ldots, 0, L-q)$ and therefore, since increasing the cutoff L by one adds just one new representation to the decomposition of $\mathcal{F}_L \otimes \mathcal{F}^*_{L-q}$, the eigenvalues of the Laplacian $\hat{\mathcal{L}}^2$ are given by $C_2(N+1, l+q, l)$ for $l = 0, \ldots, L-q$. More explicitly, a general matrix $\mathbf{M}_q$ is expanded as in (2.46), where the $\mathbf{D}$-polarization tensors form a basis for eigenvalues of the Laplacian, i.e.

$$\hat{\mathcal{L}}^2 \mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l} = \frac{1}{2}\left((l+q)(l+N) + l(l+q+N) + \frac{Nq^2}{N+1}\right) \mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l} \, , \tag{3.32}$$

for $l = 0, \ldots, n_\mathrm{R}$ (and $n_\mathrm{R} = L-q$) and are given by

$$\mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l} = \frac{\sqrt{d_N(n_\mathrm{R})}}{\sqrt{Q_N(l,q,L)}} \mathcal{P}_{\boldsymbol{\alpha}_{l+q}, \boldsymbol{\sigma}_l}{}^{\boldsymbol{\beta}_l, \boldsymbol{\tau}_{l+q}} |\boldsymbol{\tau}_{l+q}, \boldsymbol{\gamma}_{n_\mathrm{R}-l}\rangle\langle \boldsymbol{\sigma}_l, \boldsymbol{\gamma}_{n_\mathrm{R}-l}| \, , \tag{3.33}$$

which generalizes (2.47) and they are normalized such that

$$\frac{1}{d_N(n_\mathrm{R})} Tr\left((\mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l})^\dagger \mathbf{D}_{\boldsymbol{\sigma}_{l'+q'}}{}^{\boldsymbol{\tau}_{l'}} \right) = \delta_{ll'} \delta_{qq'} \mathcal{P}_{\boldsymbol{\beta}_l, \boldsymbol{\sigma}_{l+q}}{}^{\boldsymbol{\alpha}_{l+q}, \boldsymbol{\tau}_l} \tag{3.34}$$

and

$$Q_N(l,q,L) = \frac{l!(l+q)!}{(2l+N+q)!} \frac{(L-q-l)!(L+l+N)!}{(L-q)!L!} . \tag{3.35}$$

One can similarly generalize (2.51) and (2.53) with

$$|z, n_\mathrm{R}\rangle\langle n_\mathrm{L}, z| = \frac{\mathcal{T}^{1/2}_{L,q}(\hat{\mathcal{L}}^2, N)}{d_N(n_R)} \rho_{L,q}(\bar{z}, z) \tag{3.36}$$

and now the operator $\mathcal{T}_{L,q}(\hat{\mathcal{L}}^2, N)$ has eigenvalues

$$T_L(l,q,N) = \frac{d_N(n_\mathrm{R}) R_N(l,q)}{Q_N(l,q,L)} = \frac{L!(n_\mathrm{R}+N)!}{(n_\mathrm{R}-l)!(L+l+N)!} \, , \tag{3.37}$$

where

$$R_N(l,q) = \frac{N! l! (l+q)!}{(2l+N+q)!} \tag{3.38}$$

provide the coefficients in the $\mathbb{CP}^N$ generalization of (2.52).

4. Pseudo creation and annihilation operators

We would now like to obtain the generalization of the operators $\hat{K}_\pm$ for $\mathbb{CP}^N$. To this end we observe that the natural generalization of $a^{\overline{\alpha}} = a_\alpha = a^\beta \epsilon_{\beta\alpha}$ introduced in (2.39) for $su(2)$ is obtained by using the ϵ-tensor of $su(N+1)$ contracted with N oscillators. For this to be non-zero all of the oscillators need to be distinct and hence we need to introduce N sets of oscillators $a(i)^\alpha$, with $i = 1, \ldots N$. The construction will then naturally lead to dualization of the representations that occurred earlier. To avoid this we will use a set of oscillators $a^\imath_\alpha$ which when combined via the ϵ-tensor will give an N-oscillator composite operator, A^α with the same transformation properties as the a^α of the earlier section. By this route, the construction leads naturally to $a^\alpha \longmapsto A^\alpha$ and we will have merely replaced our single oscillator by a composite one. We begin with the composite operators

$$\widetilde{A}^\alpha = \widetilde{A}_{\overline{\alpha}} = \frac{1}{N!} \epsilon_{\overline{\alpha}\overline{\theta}_1\cdots\overline{\theta}_N} \epsilon_{\imath_1\cdots\imath_N} a^{\imath_1}_{\theta_1} \cdots a^{\imath_N}_{\theta_N} \tag{4.1}$$

and

$$\big(\widetilde{A}^\alpha\big)^\dagger = \widetilde{A}^\dagger_\alpha := \frac{1}{N!} \epsilon^{\imath_1\cdots\imath_N} \epsilon_{\alpha\theta_1\cdots\theta_N} (a^\dagger)^{\theta_1}_{\imath_1} \cdots (a^\dagger)^{\theta_N}_{\imath_N} \,, \tag{4.2}$$

with $\alpha_i, \beta_j = 1, \ldots, N+1$. Note that $\widetilde{A}_{\overline{\alpha}}$ reduces to (2.39) for the fuzzy sphere.

Consider the Fock space generated freely by the subset of $N(N+1)$ oscillators ${a^\dagger}^\alpha_\imath = (a^\imath_\alpha)^\dagger$, which we denote $\mathcal{F}^{\text{Total}}$. These oscillators carry the anti-fundamental representation of $u(N+1)$ and the fundamental representation of $u(N)$, with $u(N+1)$ generators

$$\hat{J}^\alpha{}_\beta = {a^\dagger}^\alpha_\imath a^\imath_\beta \tag{4.3}$$

and $u(N)$ generators

$$\hat{J}_\imath{}^\jmath = {a^\dagger}^\alpha_\imath a^\jmath_\alpha \,. \tag{4.4}$$

These have the common $u(1)$ generator

$$\hat{N} = {a^\dagger}^\alpha_\imath a^\imath_\alpha \,. \tag{4.5}$$

The generators of $u(N)$, $u(N+1)$ and $u(1)$ mutually commute, so the Fock space $\mathcal{F}^{\text{Total}}$ carries a representation of $su(N+1) \times su(N) \times \mathrm{U}(1)$ and we can decompose it into irreducible representations of these groups. Fixing on an eigenvalue of $\hat{N}$ fixes the total number of oscillators and we obtain the space $\mathcal{F}^{\text{Total}}_n$. Due to the fact that all oscillators are identical this space carries the symmetric representation of $su(N(N+1))$ and when decomposed under $su(N) \times su(N+1)$ gives a direct sum of representations. *There are no branching multiplicities in this decomposition* due to the interchangability of the identical oscillators. Furthermore, the $su(N)$ representation is sufficient to identify the spaces arising in the decomposition. When the total occupation number is NL the subspace $\mathcal{F}^{\text{Total}}_n$ contains one unique $su(n)$ singlet corresponding to L copies of $\widetilde{A}^\dagger_\alpha$ acting on the Fock vacuum and transforming under the L-fold symmetric tensor product representation of $su(N+1)$. The subspace generated freely by the $\widetilde{A}^\dagger_\alpha$, we again denote by $\mathcal{F}$ and refer to as the *reduced*

Fock space, it naturally decomposes into a direct sum of subspaces with fixed eigenvalue, L, of the "reduced number operator"

$$\hat{\mathcal{N}} = \frac{1}{N}(a^\dagger)_\imath^{\ \alpha} a^\imath_\alpha = \frac{\hat{N}}{N} \tag{4.6}$$

so that

$$\mathcal{F} = \bigoplus_{L=0}^{\infty} \mathcal{F}_L. \tag{4.7}$$

and $\hat{\mathcal{N}}$ counts the number of $\widetilde{A}^\dagger_\alpha$ acting on the Fock space vacuum. These subspaces are isomorphic to those introduced in section 3.

The reduced Fock space is orthogonal to the remainder so that we have

$$\mathcal{F}^{\text{Total}} = \mathcal{F} \oplus \mathcal{F}^\perp \ . \tag{4.8}$$

The space $\mathcal{F}^\perp$ can further be decomposed under $su(N)$ with the leading representation the anti-fundamental of $su(N)$, carried by the index $\imath$ on a single oscillator ${a^\dagger}_\imath^\alpha$. In summary we can decompose $\mathcal{F}^{\text{Total}}$ as

$$\mathcal{F}^{\text{Total}} = \oplus_{\mathcal{R}} \mathcal{F}_{\mathcal{R}} \,, \tag{4.9}$$

where the sum is over all irreducible representations $\mathcal{R}$ of $u(N)$, and due to the fact that all oscillators are identical under $u(N(N+1))$, each representation occurs precisely once in the decomposition.

The construction described here carries over to one based on oscillators $(a^\dagger)_I^\alpha$ carrying representations of $u(N+1)$ and $u(k)$ with $k \leq N$. The resulting Fock space is then decomposed as in (4.9) with the sum over irreducible representations of $u(k)$ and again there are no multiplicities as the symmetric representations of $u((N+1)k)$ break up into a direct sum of representations of $u(N+1) \otimes u(k)$ without degeneracy and the decomposition can be labeled by the representations of $u(k)$, the smaller group.

Given that we have identified the spaces $\mathcal{F}_L$, then $\mathcal{F} \otimes \mathcal{F}^*$ provides the matrix algebra. The principal distinction is that now *a composite oscillator plays the role of the single oscillator*. However, for $N \geq 2$, the $\widetilde{A}^\dagger_\alpha$ and $\widetilde{A}^\alpha$ do not satisfy the Heisenberg algebra, e.g. for $N = 2$

$$[\widetilde{A}^\beta, \widetilde{A}^\dagger_\alpha] = 2\delta_\alpha^{\ \beta}\left(1 + \hat{\mathcal{N}}\right) - (a^\dagger)^\beta_\imath a^\imath_\alpha = 2\delta_\alpha^{\ \beta}\left(1 + \hat{\mathcal{N}}\right) - \hat{J}^\beta_{\ \ \alpha} \tag{4.10}$$

with

$$\hat{\mathcal{N}} := \frac{1}{2}(a^\dagger)^\delta_\imath a^\imath_\delta \ . \tag{4.11}$$

However, as we will see, it is straightforward to modify $\widetilde{A}^\alpha$ and $\widetilde{A}^\dagger_\alpha$ to get a more suitable algebra, where they do satisfy the Heisenberg algebra on a reduced Fock space generated by the $\widetilde{A}^\dagger_\alpha$. We shall refer to $A^\dagger_\alpha$ and A^α as pseudo creation and pseudo annihilation operators and the subspace of the full Fock space spanned by $|\boldsymbol{\alpha}\rangle$ is what we have termed the reduced Fock space. Note: The reduced Fock space is the same as the full Fock space only for $N = 1$.

Let us examine the action of $\widetilde{A}^\alpha$ on $\mathcal{F}_L$. Define

$$|\,\widetilde{\boldsymbol{\alpha}}\,\rangle := \frac{1}{L!}\widetilde{A}^\dagger_{\alpha_1} \cdots \widetilde{A}^\dagger_{\alpha_L}\,|0\rangle \tag{4.12}$$

and

$$|\,\widetilde{\boldsymbol{\alpha}}_k\,\rangle := \frac{1}{(L-1)!}\widetilde{A}^\dagger_{\alpha_1}\cdots\widetilde{A}^\dagger_{\alpha_{k-1}}\widetilde{A}^\dagger_{\alpha_{k+1}}\widetilde{A}^\dagger_{\alpha_L}\,|\,0\,\rangle \tag{4.13}$$

where $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_L)$ and $\boldsymbol{\alpha}_k = (\alpha_1,\ldots,\alpha_{k-1},\alpha_{k+1},\ldots,\alpha_L)$. Then (see appendix C) for $L \geq 1$,

$$\widetilde{A}^\beta\,|\,\widetilde{\boldsymbol{\alpha}}\,\rangle = c_N(L)\sum_{i=1}^{L}\delta^\beta_{\alpha_i}|\,\widetilde{\boldsymbol{\alpha}}_i\,\rangle \tag{4.14}$$

with

$$c_N(L) = \frac{(N+L-1)!}{L!}. \tag{4.15}$$

This suggests re-normalizing $\widetilde{A}^\alpha$:

$$\widetilde{A}^\alpha \longrightarrow A^\alpha := \widetilde{A}^\alpha\frac{1}{\sqrt{c_N(\hat{\mathcal{N}})}} \qquad \widetilde{A}^\dagger_\alpha \longrightarrow A^\dagger_\alpha := \frac{1}{\sqrt{c_N(\hat{\mathcal{N}})}}\widetilde{A}^\dagger_\alpha\,, \tag{4.16}$$

so that

$$A^\beta\,|\,\boldsymbol{\alpha}\,\rangle = \sum_{i=1}^{l}\delta^\beta_{\alpha_i}\,|\,\boldsymbol{\alpha}_i\,\rangle \tag{4.17}$$

and

$$[A^\alpha,\,A^\dagger_\beta]\,|\,\boldsymbol{\gamma}\,\rangle = \delta^\alpha_\beta\,|\,\boldsymbol{\gamma}\,\rangle\,, \tag{4.18}$$

where $|\,\boldsymbol{\alpha}\,\rangle$ is $|\,\widetilde{\boldsymbol{\alpha}}\,\rangle$ with $\widetilde{A}^\dagger_{\alpha_k}$ replaced with $A^\dagger_{\alpha_k}$, etc. Now (4.18) implies that $A^\dagger_\alpha$ and A^α *act as simple creation and annihilation operators* on $\mathcal{F}$, the subspace of singlet representations of $\mathcal{F}^{\mathrm{Total}}$. For $\mathcal{F}$ we further have that

$$\hat{\mathcal{N}}_A\mathcal{F} = \hat{\mathcal{N}}\mathcal{F} \tag{4.19}$$

where $\hat{\mathcal{N}}_A = A^\dagger_\alpha A^\alpha$, so the two number operators agree on the reduced Fock space. Of course $A^\dagger_\alpha$ and A^β do not satisfy a Heisenberg algebra on the whole Fock space $\mathcal{F}^{\mathrm{Total}}$.

The spaces $\mathcal{F}_L$ then can be identified as the space of vectors $|\,\boldsymbol{\alpha}\,\rangle$ satisfying the relations

$$\hat{J}_\imath{}^\jmath\,|\,\boldsymbol{\alpha}\,\rangle = \delta^\jmath_\imath\hat{\mathcal{N}}\,|\,\boldsymbol{\alpha}\,\rangle = L\delta^\jmath_\imath\,|\,\boldsymbol{\alpha}\,\rangle \quad \text{and} \quad \hat{\mathcal{N}}_A\,|\,\boldsymbol{\alpha}\,\rangle = L\,|\,\boldsymbol{\alpha}\,\rangle \tag{4.20}$$

the first indicating the singlet nature of the state under $u(N)$.

As before, a basis for $\mathcal{F}_{n_\mathrm{L}}\otimes\mathcal{F}^*_{n_\mathrm{R}}$ is given by $|\,\boldsymbol{\alpha}\,\rangle\,\langle\,\boldsymbol{\beta}\,|$ for all possible strings $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_{n_\mathrm{L}})$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_{n_\mathrm{R}})$ and the constructions of the previous section are unchanged. As in (3.6), a general matrix takes the form

$$\mathbf{M} = \frac{1}{L!}M^{\alpha_1\cdots\alpha_L}{}_{\beta_1\cdots\beta_L}A^\dagger_{\alpha_1}\cdots A^\dagger_{\alpha_L}\,|\,0\,\rangle\,\langle\,0\,|\,A^{\beta_1}\cdots A^{\beta_L}. \tag{4.21}$$

where the only distinction is that the oscillators a^α are replaced by A^α. The derivatives become

$$\hat{\boldsymbol{\mathcal{L}}}_a = \left(A^\dagger\frac{\lambda_a}{2}A\right)^{\mathrm{L}} - \left(A^\dagger\frac{\lambda_a}{2}A\right)^{\mathrm{R}} \tag{4.22}$$

and the Laplacian is now replaced by $\hat{\mathcal{L}}^2$ as in section 2 which fixes the geometry to be that of section 3.

Had we proceeded with a set of oscillators $(a^\dagger)^{\imath}_{\alpha}$, there would be the minor consequence that $\mathcal{F}_L$ would transform as the L-fold symmetric tensor product of the anti-fundamental rather than of the fundamental, i.e. $\mathcal{F}_L$ would carry the representation $(0,0,\ldots,L)$ and hence the coefficients in the matrix algebra $M^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$ would be replaced by $M_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$, so that the role of $\mathbf{M}$ would be replaced by that of $\mathbf{M}^\dagger$.

5. Realization of the $\hat{K}_\imath$ and $\hat{K}_{\bar\imath}$

The natural generalization of (2.41) is now clear and we can define:

$$\begin{aligned}
\hat{K}_\imath &:= \left(A^\dagger_\alpha\right)^{\mathrm{L}}\left((a^\dagger)^\alpha_\imath\right)^{\mathrm{R}} &&: \mathcal{F}_L\otimes\mathcal{F}^*_L \longmapsto \mathcal{F}_{L+1}\otimes\mathcal{F}^*_{L-1,\imath}\\
\hat{K}_{\bar\imath} &:= \quad (A^\alpha)^{\mathrm{L}}(a^\imath_\alpha)^{\mathrm{R}} &&: \mathcal{F}_L\otimes\mathcal{F}^*_L \longmapsto \mathcal{F}_{L-1}\otimes\mathcal{F}^*_{L,\bar\imath}\\
\hat{K}_0 &:= \tfrac{1}{2}(\hat{\mathcal{N}}^{\mathrm{L}}_A-\hat{\mathcal{N}}^{\mathrm{R}}) &&: \mathcal{F}_L\otimes\mathcal{F}^*_L \longmapsto 0\,,
\end{aligned}\tag{5.1}$$

where $\hat{\mathcal{N}}_A = A^\dagger_\alpha A^\alpha$ and we have denoted the subspace of Fock space spanned by vectors of the form

$$(a^\dagger)^\alpha_\imath A^\dagger_{\alpha_1}\ldots A^\dagger_{\alpha_L}\,|0\rangle \tag{5.2}$$

by $\mathcal{F}_{L,\imath}=\mathcal{F}_L{}^{\bar\imath}$. Note: Since $(a^\dagger)^\alpha_\imath A^\dagger_\alpha=0$ all contractions between the $a^\imath_\alpha$ and the A^α vanish.

Observe that, quite generally, $[\hat{K}_\imath,\hat{K}_\jmath]=0=[\hat{K}_{\bar\imath},\hat{K}_{\bar\jmath}]$ and for a generic non-square matrix $\mathbf{M}_q\in\mathcal{F}_{n_{\mathrm{L}}}\otimes\mathcal{F}^*_{n_{\mathrm{R}}}$ with $q=n_{\mathrm{L}}-n_{\mathrm{R}}$, using equations (4.18) and (4.20) we can evaluate the commutator

$$\begin{aligned}
[\hat{K}_\imath,\,\hat{K}_{\bar\jmath}]\mathbf{M}_q &= A^\dagger_\alpha A^\beta\mathbf{M}_q a^\jmath_\beta(a^\dagger)^\alpha_\imath - A^\beta A^\dagger_\alpha\mathbf{M}_q(a^\dagger)^\alpha_\imath a^\jmath_\beta\\
&= A^\dagger_\alpha A^\beta\mathbf{M}_q[a^\jmath_\beta,(a^\dagger)^\alpha_\imath] - \mathbf{M}_q\hat{J}_\imath{}^\jmath\\
&= 2\delta^\jmath_\imath\hat{K}_0\mathbf{M}_q = q\delta^\jmath_\imath\mathbf{M}_q\ .
\end{aligned}\tag{5.3}$$

So we have

$$[\hat{K}_\imath,\,\hat{K}_{\bar\jmath}]\mathbf{M}_q = 2\delta^{\bar\jmath}_\imath\hat{K}_0\mathbf{M}_q\ . \tag{5.4}$$

In particular the algebra $\mathcal{F}_L\otimes\mathcal{F}^*_L$ is annihilated by $[\hat{K}_\imath,\hat{K}_{\bar\jmath}]$, as expected for the action of the holonomy group on functions.

The Laplacian, Δ_K, acting on $\mathbf{M}_q$ constructed from the $K_\imath$ and $K_{\bar\imath}$, as demonstrated below, is naturally given by

$$\Delta_K = \frac{1}{2}\left(\hat{K}_\imath\hat{K}_{\bar\imath}+\hat{K}_{\bar\imath}\hat{K}_\imath\right)+\frac{2N}{N+1}\hat{K}_0^2\ . \tag{5.5}$$

A little computation demonstrates that we can re-express Δ_K in the form

$$\Delta_K = (\hat{L}^{\mathrm{L}}_a-\hat{J}^{\mathrm{R}}_a)^2\,, \tag{5.6}$$

where $\hat{L}_a = A^\dagger\frac{\lambda_a}{2}A$ and $\hat{J}_a=(a^\dagger)_\imath\frac{\overline{\lambda}_a}{2}a^\imath$. Now it is easy to verify that for states $|\boldsymbol{\gamma}\rangle\in\mathcal{F}$

$$\hat{L}_a\,|\boldsymbol{\gamma}\rangle = \hat{J}_a\,|\boldsymbol{\gamma}\rangle \tag{5.7}$$

and hence we have for $\mathbf{M}_{\mathcal{R}} \in \mathcal{F}_L \otimes \mathcal{F}^*_{\mathcal{R}}$ where $\mathcal{R}$ is any irreducible representation of $u(N)$,

$$\Delta_K \mathbf{M}_{\mathcal{R}} = (\hat{L}^{\rm L}_a - \hat{J}^{\rm R}_a)^2 \mathbf{M}_{\mathcal{R}} = (\hat{J}^{\rm L}_a - \hat{J}^{\rm R}_a)^2 \mathbf{M}_{\mathcal{R}}\,, \tag{5.8}$$

where $(\hat{J}^{\rm L}_a - \hat{J}^{\rm R}_a)^2$ is the $su(N+1)$ quadratic Casimir and provides the natural Laplacian on the entire space $\mathcal{F}^{\rm Total} \otimes \mathcal{F}^{\rm Total *}$. The eigenspaces of Δ_K are the irreducible representations in the decomposition of $\mathcal{F}_L \otimes \mathcal{F}^*_{\mathcal{R}}$ and the eigenvalues are those of the $su(N+1)$ quadratic Casimir in this representation.

Observe also that we can further generalize (5.4) to obtain

$$[\hat{K}_\imath,\, \hat{K}_{\bar{\jmath}}]\mathbf{M}_{\mathcal{R}} = -\mathbf{M}_{\mathcal{R}} \hat{J}_\imath{}^\jmath + \delta^\jmath_\imath A^\dagger_\alpha A^\alpha \mathbf{M}_{\mathcal{R}} = -\mathbf{M}_{\mathcal{R}}(\hat{J}_\imath{}^\jmath - \delta^\jmath_\imath \hat{\mathcal{N}}) + 2\delta^\jmath_\imath \hat{K}_0 \mathbf{M}_{\mathcal{R}}\,, \tag{5.9}$$

where $\hat{J}_\imath{}^\jmath - \delta^\jmath_\imath \hat{\mathcal{N}}$ are the $su(N)$ generators and $\hat{K}_0$ is the $u(1)$ generator introduced above. Hence the effect of the commutator action $[\hat{K}_\imath,\, \hat{K}_{\bar{\jmath}}]$ on $\mathbf{M}_{\mathcal{R}}$ is that of a generator of the representation $\mathcal{R}$ and it effects a rotation in $\mathcal{R}$.

To obtain the spectrum of Δ_K by direct computation, observe that

$$\hat{K}_\imath \,|\boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}| = 0 \quad \text{and} \quad \hat{K}_{\bar{\imath}} \,|\boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}| = 0\,, \tag{5.10}$$

which can be established by acting on $|\beta, \boldsymbol{\gamma}\rangle = \frac{1}{\sqrt{L!}} A^\dagger_\beta A^\dagger_{\gamma_1} \cdots A^\dagger_{\gamma_L}\,|0\rangle$,

$$\begin{aligned} \hat{K}_\imath \,|\beta, \boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}, \beta| &= A^\dagger_\delta A^\dagger_\beta \,|\boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}|\,[A^\beta,\, (a^\dagger)^\delta_i] + A^\dagger_\beta (\hat{K}_\imath \,|\boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}|) A^\beta \\ &= A^\dagger_\beta (\hat{K}_\imath \,|\boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}|) A^\beta\ . \end{aligned}$$

using antisymmetry in equation (C.3) of appendix C and $\hat{K}_\imath \,|0\rangle\,\langle 0| = 0$.

We can be more general and consider a non-square matrix made from a single polarization tensor, i.e. $\boldsymbol{\Phi}_{l_L, l_R} = f^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} \,|\boldsymbol{\alpha}_{l_L}, \boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}, \boldsymbol{\beta}_{l_R}|$ where the coefficients $f^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$ are completely traceless and the indices $\boldsymbol{\gamma}_k$, with $k = 1, \ldots, n_{\rm L} - l_L = n_{\rm R} - l_R$ contracted. Then

$$\hat{K}_\imath \,|\boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}, \boldsymbol{\beta}| = A^\dagger_\mu A^\dagger_{\alpha_1} \cdots A^\dagger_{\alpha_{l_L}} \,|\boldsymbol{\gamma}\rangle\,\langle\boldsymbol{\gamma}|\,[A^{\beta_1} \cdots A^{\beta_{l_R}},\, (a^\dagger)^\mu_i]\ . \tag{5.11}$$

Using (C.7) from appendix C and (4.17), we now find

$$\begin{aligned} \hat{K}_{\bar{\imath}} \hat{K}_\imath \boldsymbol{\Phi}_{l_L, l_R} &= \sum_{k=1}^{l_R} f^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} A^\nu A^\dagger_\mu \,|\boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle \left(\langle\boldsymbol{\gamma}, \boldsymbol{\beta}|\,\delta^\mu_\nu - \langle\boldsymbol{\gamma}, \hat{\boldsymbol{\beta}}_k|\,A^\mu \delta^{\beta_k}_\nu\right) \\ &= l_R(N + n_{\rm L} + 1)\boldsymbol{\Phi}_{l_L, l_R} - \sum_{k=1}^{l_R} f^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} \left(\delta^{\beta_k}_\mu \,|\boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle + \sum_{i=1}^{n_{\rm L} - l_L} \delta^{\beta_k}_{\gamma_i} \,|\mu, \boldsymbol{\alpha}, \hat{\boldsymbol{\gamma}}_i\rangle\right) \langle\boldsymbol{\gamma}, \hat{\boldsymbol{\beta}}_k, \mu| \\ &= l_R(l_L + N)\boldsymbol{\Phi}_{l_L, l_R}. \end{aligned} \tag{5.12}$$

Similarly,

$$\hat{K}_\imath \hat{K}_{\bar{\imath}} \boldsymbol{\Phi}_{l_L, l_R} = l_L(l_R + N)\boldsymbol{\Phi}_{l_L, l_R}\ . \tag{5.13}$$

In particular for square matrices with $n_{\rm L} = n_{\rm R} = L$ and $l_l = l_R = l$,

$$\frac{1}{2}\big(\hat{K}_\imath \hat{K}_{\bar{\imath}} + \hat{K}_{\bar{\imath}} \hat{K}_\imath\big)\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l} = l(l + N)\mathbf{Y}_{\boldsymbol{\alpha}_l}{}^{\boldsymbol{\beta}_l}, \tag{5.14}$$

where the polarization tensors are now built, as before, by projecting onto an irreducible representation of $su(N+1)$, but states are built with the $A^\dagger_\alpha$.

For non-square matrices we can use the polarization tensors (again built as before) $\mathbf{D}_{\boldsymbol{\alpha}_{l+q}}{}^{\boldsymbol{\beta}_l}$ and with the Laplacian (5.5) we have

$$\Delta_K \mathbf{D}_{\boldsymbol{\alpha}_{l_L}}{}^{\boldsymbol{\beta}_{l_R}} = \frac{1}{2}\left(l_L(l_R+N) + l_R(l_L+N) + \frac{N(l_L-l_R)^2}{(N+1)}\right)\mathbf{D}_{\boldsymbol{\alpha}_{l_L}}{}^{\boldsymbol{\beta}_{l_R}} . \tag{5.15}$$

Hence putting $l_R = l$ and $q = l_L - l_R$ we agree with the spectrum (3.32) as derived earlier in terms of the one oscillator formulation.

Finally we can map our modules to sections of equivariant vector bundles tensored with functions[8] using either $\rho_{L,q}(\bar{z},z)$ or $|z,n_{\mathrm{R}}\rangle\langle n_{\mathrm{L}},z|$ now built from pseudo creation and annihilation operators. These will induce the mapping

$$\begin{aligned}
&\left(\frac{1}{\sqrt{\hat{\mathcal{N}}_A}}A^\dagger_\alpha\right)^{\mathrm{L}} \longmapsto \bar{z}_\alpha \qquad \left(A^\alpha\sqrt{\hat{\mathcal{N}}_A}\right)^{\mathrm{L}} \longmapsto \frac{\partial}{\partial \bar{z}_\alpha}\\
&\left(A^\alpha\frac{1}{\sqrt{\hat{\mathcal{N}}_A}}\right)^{\mathrm{R}} \longmapsto z^\alpha \qquad \left(\sqrt{\hat{\mathcal{N}}_A}A^\dagger_\alpha\right)^{\mathrm{R}} \longmapsto \frac{\partial}{\partial z^\alpha}
\end{aligned} \tag{5.16}$$

as expected. For elements of more general modules, such as $\mathbf{M}_{\mathcal{R}}$, one needs to introduce commutative analogues of the $a^\imath_\alpha$. For these, we will introduce $u^\imath_\alpha$ (see appendix B). By following analogous constructions to those above, we can induce maps

$$\begin{aligned}
((a^\dagger)^\alpha_\imath)^{\mathrm{L}} \longmapsto u^\alpha_\imath \qquad &\text{and} \qquad (a^\imath_\alpha)^{\mathrm{R}} \longmapsto \bar{u}^\imath_\alpha\\
(a^\imath_\alpha)^{\mathrm{L}} \longmapsto \frac{\partial}{\partial u^\alpha_\imath} \qquad &\text{and} \quad ((a^\dagger)^\alpha_\imath)^{\mathrm{R}} \longmapsto \frac{\partial}{\partial \bar{u}^\imath_\alpha}
\end{aligned} \tag{5.17}$$

modulo normalizations. Using appendix B we can identify the image of the operators.

Taking the adjoint of the $\mathbf{M}_{\mathcal{R}}$ gives us ${}_{\overline{\mathcal{R}}}\mathbf{M}^\dagger \in \mathcal{F}_{\overline{\mathcal{R}}} \otimes \mathcal{F}^*_L$. The natural operators acting on these modules are ${}_\imath\hat{K}$ and ${}_{\bar{\imath}}\hat{K}$ which are the adjoints of $\hat{K}_{\bar{\imath}}$ and $\hat{K}_\imath$ respectively. Explicitly they are:

$${}_\imath\hat{K} = ((a^\dagger)^\alpha_\imath)^{\mathrm{L}}(A^\dagger_\alpha)^{\mathrm{R}} = (\hat{K}_{\bar{\imath}})^\dagger \quad \text{and} \quad {}_{\bar{\imath}}\hat{K} = (a^\imath_\alpha)^{\mathrm{L}}(A^\alpha)^{\mathrm{R}} = (\hat{K}_\imath)^\dagger \tag{5.18}$$

One can then see that the pairings we consider to build our Laplacians arise naturally from action functionals of the form

$$\frac{Tr}{d_N(\mathcal{R})}\left((\hat{K}_\imath\mathbf{\Psi}_{\mathcal{R}})^\dagger\hat{K}_\imath\mathbf{\Phi}_{\mathcal{R}} + (\hat{K}_{\bar{\imath}}\mathbf{\Psi}_{\mathcal{R}})^\dagger\hat{K}_{\bar{\imath}}\mathbf{\Phi}_{\mathcal{R}}\right) \tag{5.19}$$

or

$$\frac{Tr}{d_N(n_{\mathrm{L}})}\left(\hat{K}_\imath\mathbf{\Psi}_{\mathcal{R}}(\hat{K}_\imath\mathbf{\Phi}_{\mathcal{R}})^\dagger + \hat{K}_{\bar{\imath}}\mathbf{\Psi}_{\mathcal{R}}(\hat{K}_{\bar{\imath}}\mathbf{\Phi}_{\mathcal{R}})^\dagger\right) \tag{5.20}$$

where $\mathbf{\Psi}_{\mathcal{R}}$ and $\mathbf{\Phi}_{\mathcal{R}}$ are elements of the module $\mathcal{F}_L \otimes \mathcal{F}^*_{\mathcal{R}}$. We will leave the discussion of action functionals and field theories to a separate publication.

[8] An alternative quantization of equivariant vector bundles using Toeplitz quantization can be found in [47–49].

6. Conclusions

In this paper, we have re-examined the construction of $\mathbb{CP}^N$. The principal novelties of the current work are:

- The introduction of the double vacuum representation which is used to obtain the modules representing noncommutative complex line bundles. The double vacuum, $|0\rangle\langle 0|$ is inserted, between the creation and annihilation operators of a normal ordered homogeneous polynomial in creation and annihilation operators.
- A construction of polarization tensors that maintains equivariance at every step and renders the polarization tensors for non-square matrices very tractable and can be readily generalized to a large class of spaces.
- We simplified the Laplacians acting on noncommutative line bundles.
- We introduced a symmetric symbol density to replace the coherent state map. This density should have an interest beyond the current work and raises questions for the future: Does this density map positive matrices to positive functions as the coherent state map does?
- We found a new Fock space construction of $\mathbb{CP}^N$. The construction allows us to access all equivariant complex vector bundles over $\mathbb{CP}^N$ which are in one to one correspondence with the representation ring of $\mathrm{U}(N)$ (see 237 H of the Encyclopedic Dictionary of Mathematics, second edition [50]).
- We have found composite oscillators, A^α of eq. (4.16), that obey the Heisenberg algebra on the reduced Fock space freely generated by these oscillators.
- Along the way we have found a natural generalization to $su(N)$ of the Schwinger-Jordan construction for $su(2)$ that avoids multiplicities and has resonance with the work of Chaturvedi et al [51, 52].

The work described here opens many additional directions for investigation. The double Fock vacuum reformulation of $\mathbb{CP}^N$ has already led to a simplified version of the star product [42]. The general approach taken here leads naturally to the construction of all flag and superflag manifolds [53], where the algebra of functions is given by $\mathcal{F}_{\mathcal{R}} \otimes \mathcal{F}^*_{\overline{\mathcal{R}}}$ with for example the representation $\mathcal{R}$ of $u(k)$ on the left and the conjugate representation $\overline{\mathcal{R}}$ on the right realizing the algebra for the Grassmanian $\mathbb{G}\mathrm{r}_{\mathrm{k;N+1}}$.

Once it is appreciated that single oscillators can be replaced by composites, many new possibilities emerge with consequences far beyond the current work. For example, the construction should prove useful in describing quasi-holes and quasi-particles in the higher dimensional quantum Hall effect [54–56, 15]. Closer to home, within the framework of noncommutative geometry, for example, one can build additional structure into the Moyal plane by taking determinant composite oscillators, A, of $su(N)$, where

$$A = \det(a)\sqrt{\frac{\hat{\mathcal{N}}!}{(\hat{\mathcal{N}}+N-1)!}} \tag{6.1}$$

and a is the matrix of oscillators a_I^α and $\hat{\mathcal{N}}$ is the reduced number operator

$$\hat{\mathcal{N}} = \frac{(a^\dagger)_\alpha^I a_I^\alpha}{N} \ .$$

Many further generalizations are possible. The most obvious next step is the construction of spinor fields and their associated action functionals. We will return to this in a subsequent article [57].

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A. $su(2)$ derivatives on S^2

For notational convenience we label the columns and rows of the entries in the matrix differently, as u_I^α with $I = 0, 1$ and $\alpha = 1, 2$, for future convenience. Thus

$$U = \begin{pmatrix} u_0^1 & u_1^1 \\ u_0^2 & u_1^2 \end{pmatrix} \in \mathrm{SU}(2). \tag{A.1}$$

Of course not all four entries are independent, we can write U as[9]

$$U = \begin{pmatrix} z^1 & -\bar{z}_2 \\ z^2 & \bar{z}_1 \end{pmatrix}, \tag{A.2}$$

where z^α, satisfying $z^\dagger z = 1$, label points on S^3 and project to coordinates on S^2. An alternative parameterisation is

$$U = \begin{pmatrix} \bar{u}_2^1 & u_1^1 \\ -\bar{u}_1^1 & u_1^2 \end{pmatrix} \tag{A.3}$$

with $u_0^\alpha = \epsilon^{\alpha\beta}\bar{u}_\beta^1$, where $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ with $\epsilon^{12} = 1$.

There are three linearly independent left vector fields on $\mathrm{SU}(2) \cong S^3$ at U as well as three linearly independent right vector fields, which we can choose to be generated by $\frac{\sigma_a}{2}$ where σ_a are the Pauli matrices, $a = 1, 2, 3$,

$$\mathcal{L}_a(U) = -\left(\frac{\sigma_a}{2}\right) U \qquad \mathcal{R}_a(U) = U \left(\frac{\sigma_a}{2}\right). \tag{A.4}$$

It is straightforward to write these as differential operators

$$-\left(\frac{\sigma_a}{2}\right) U = -\frac{1}{2}\left(u_I^\beta (\sigma_a)^\alpha{}_\beta \frac{\partial}{\partial u_I^\alpha}\right) U, \quad U\left(\frac{\sigma_a}{2}\right) = \frac{1}{2}\left(u_I^\alpha (\sigma_a)^I{}_J \frac{\partial}{\partial u_J^\alpha}\right) U, \tag{A.5}$$

[9] Indices are raised and lowered using the unitary metric on $\mathbb{C}^2$, $\delta^{\alpha\bar{\beta}}$ or $\delta_{\alpha\bar{\beta}}$. Thus $\overline{(z^\alpha)} = (\bar{z})_\alpha = \bar{z}_\alpha$. Similarly $\overline{u_I^\alpha} = \bar{u}_I^{\bar{\alpha}} = \bar{u}_\alpha^I$.

where the partial derivatives are taken as though the u_I^α were independent.[10] These differential operators satisfy the $su(2)$ algebra

$$[\mathcal{L}_a, \mathcal{L}_b] = i\epsilon_{ab}{}^c\mathcal{L}_c, \qquad [\mathcal{R}_a, \mathcal{R}_b] = i\epsilon_{ab}{}^c\mathcal{R}_c. \tag{A.6}$$

In the alternative raising and lowering basis (2.6) where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and $\sigma_0 = \frac{1}{2}\sigma_3$, the right invariant vector fields can be written, using (z^1, z^2), as

$$\begin{aligned}
\mathcal{L}_+ &= -z^2\frac{\partial}{\partial z^1} + \bar{z}_1\frac{\partial}{\partial \bar{z}_2} \\
\mathcal{L}_- &= -z^1\frac{\partial}{\partial z^2} + \bar{z}_2\frac{\partial}{\partial \bar{z}_1} \\
\mathcal{L}_0 &= -\frac{1}{2}\left(z^1\frac{\partial}{\partial z^1} - z^2\frac{\partial}{\partial z^2}\right) + \frac{1}{2}\left(\bar{z}_1\frac{\partial}{\partial \bar{z}_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2}\right).
\end{aligned} \tag{A.7}$$

These project trivially from S^3 to the three linearly dependent Killing vector fields on S^2, since they are invariant under a phase change $z^\alpha \to e^{i\phi}z^\alpha$. Note that (A.7) decompose into two mutually commuting copies of the SU(2) algebra,

$$\mathcal{L}_+ = L_+ - \overline{L_-} \qquad \mathcal{L}_- = L_- - \overline{L_+} \qquad \mathcal{L}_0 = L_0 - \overline{L_0}, \tag{A.8}$$

where

$$L_+ = -z^2\frac{\partial}{\partial z^1}, \qquad L_- = -z^1\frac{\partial}{\partial z^2}, \qquad L_0 = -\frac{1}{2}\left(z^1\frac{\partial}{\partial z^1} - z^2\frac{\partial}{\partial z^2}\right), \tag{A.9}$$

associated with holomorphic and anti-holomorphic vector fields on $\mathbb{CP}^1$. The right vector fields on SU(2)

$$\begin{aligned}
\mathcal{R}_+ &= -z^1\frac{\partial}{\partial \bar{z}_2} + z^2\frac{\partial}{\partial \bar{z}_1} = -\epsilon^{\alpha\beta}\,\bar{u}^1_\alpha\frac{\partial}{\partial u_1^\beta} \\
\mathcal{R}_- &= -\bar{z}_2\frac{\partial}{\partial z^1} + \bar{z}_1\frac{\partial}{\partial z^2} = \epsilon_{\alpha\beta}u_1^\alpha\frac{\partial}{\partial \bar{u}^1_\beta} \\
\mathcal{R}_0 &= \frac{1}{2}\left(z^\alpha\frac{\partial}{\partial z^\alpha} - \bar{z}_\alpha\frac{\partial}{\partial \bar{z}_\alpha}\right) = \frac{1}{2}\left(\bar{u}^1_\alpha\frac{\partial}{\partial \bar{u}^1_\alpha} - u_1^\alpha\frac{\partial}{\partial u_1^\alpha}\right)
\end{aligned} \tag{A.10}$$

do not project to vector fields on S^2, rather $\mathcal{R}_+$ maps functions to $(0,1)$ tensors and $\mathcal{R}_-$ maps functions to $(1,0)$ tensors, while $\mathcal{R}_0$ generates tangent space rotations at any point on S^2 that leave functions invariant, i.e. it generates the holonomy group U(1) at the point z^α of S^2. Nevertheless the Laplacian acting on functions on $\mathbb{CP}^1$ can be written either in terms of the left or the right vector fields,

$$\begin{aligned}
-\nabla^2 &= \frac{1}{2}\left(\mathcal{L}_+\mathcal{L}_- + \mathcal{L}_-\mathcal{L}_+\right) + \mathcal{L}_0^2 = \frac{1}{2}\left(\mathcal{R}_+\mathcal{R}_- + \mathcal{R}_-\mathcal{R}_+\right) + \mathcal{R}_0^2 \\
&= -\partial.\bar{\partial} + \frac{1}{2}(z.\partial)(\bar{z}.\bar{\partial}) + \frac{1}{4}(z.\partial)(z.\partial + 2) + \frac{1}{4}(\bar{z}.\bar{\partial})(\bar{z}.\bar{\partial} + 2),
\end{aligned} \tag{A.11}$$

where $z.\partial = z^\alpha\frac{\partial}{\partial z^\alpha}$.

[10] In our notation, the entries of σ_a are necessarily labeled differently for the right and left invariant vector fields. This is an advantage of the notation in that it is clear from the index structure which is left and which is right acting.

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B. Left and right $\mathrm{SU}(N+1)$ invariant derivations on $\mathbb{CP}^N$

Once we have mapped our modules to a representation in terms of z^α and $u^\alpha_\imath$ we have the following realization.

A parameterisation of an element U of $G = \mathrm{SU}(N+1)$ can be given by

$$U = \begin{pmatrix} u^1_0 & u^1_1 & \cdots & u^1_N \\ u^2_0 & u^2_1 & \cdots & u^2_N \\ \vdots & \vdots & \vdots & \vdots \\ u^{N+1}_0 & u^{N+1}_1 & \cdots & u^{N+1}_N \end{pmatrix} \tag{B.1}$$

where

$$\bar{u}^I_\alpha u^\beta_I = \delta^\beta_\alpha, \quad \bar{u}^I_\alpha \cdot u^\alpha_J = \delta^I{}_J \quad \text{and} \quad \epsilon_{\alpha_1\cdots\alpha_{N+1}} u^{\alpha_1}_0 u^{\alpha_2}_1 \cdots u^{\alpha_{N+1}}_N = 1 \tag{B.2}$$

with $I, J = 0, 1, \ldots N$ and $\alpha_r = 1, \ldots, N+1$ for $r = 1, \ldots, N+1$.

Acting with $G = \mathrm{SU}(N+1)$ on $\mathrm{SU}(N+1)$ we find that the induced left and right vector fields are given by

$$\mathcal{L}_a = -\frac{1}{2} u^\beta_I (\lambda_a)^\alpha{}_\beta \frac{\partial}{\partial u^\alpha_I} \quad \text{and} \quad \mathcal{R}_a = \frac{1}{2} u^\alpha_I (\lambda_a)^I{}_J \frac{\partial}{\partial u^\alpha_J}\,, \tag{B.3}$$

respectively, where $\frac{\lambda_a}{2}$ are the generators of $\mathrm{SU}(N+1)$ in the fundamental representation. An alternative basis, using raising and lowering operators and the Cartan subalgebra, is obtained from the completeness relation (3.9) giving

$$\mathcal{L}^\alpha{}_\beta = -u^\alpha_I \frac{\partial}{\partial u^\beta_I} + \left(\frac{\delta^\alpha{}_\beta}{N+1}\right) u^\gamma_I \frac{\partial}{\partial u^\gamma_I} \tag{B.4}$$

$$\mathcal{R}^I{}_J = u^\alpha_J \frac{\partial}{\partial u^\alpha_I} - \left(\frac{\delta^I{}_J}{N+1}\right) u^\alpha_K \frac{\partial}{\partial u^\alpha_K}. \tag{B.5}$$

The differential operators $\mathcal{L}^\alpha{}_\beta$ and $\mathcal{R}^I{}_J$ separately satisfy the commutation relations of $su(N+1)$, without recourse to (B.2), and commute with each other.

The coset space $\mathrm{U}(N+1)/\mathrm{U}(N) = S^{2N+1}$ can be realized by embedding $\mathrm{U}(N)$ in $\mathrm{U}(N+1)$ as

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & h \end{pmatrix} \tag{B.6}$$

and taking coordinates on S^{2N+1} as the first column of (B.1). Restricting to functions $x_a = \bar{z}\lambda_a z$ projects S^{2N+1} to functions on $\mathbb{CP}^N$. Special unitarity of U implies that

$$z^\alpha := u^\alpha_0 = \epsilon^{\alpha\beta_1\cdots\beta_N} \bar{u}^1_{\beta_1} \cdots \bar{u}^N_{\beta_N} = \frac{1}{N!} \epsilon^{\alpha\beta_1\cdots\beta_N} \epsilon_{\imath_1\cdots\imath_N} \bar{u}^{\imath_1}_{\beta_1} \cdots \bar{u}^{\imath_N}_{\beta_N} \tag{B.7}$$

and is consistent with (B.2).

An equivalent description is to use

$$\overline{z^\alpha} = (\bar{z})_\alpha := \bar{u}^0_\alpha = \epsilon_{\alpha\beta_1\cdots\beta_N} u^{\beta_1}_1 \cdots u^{\beta_N}_N = \frac{1}{N!} \epsilon_{\alpha\beta_1\cdots\beta_N} \epsilon^{\imath_1\cdots\imath_N} u^{\beta_1}_{\imath_1} \cdots u^{\beta_N}_{\imath_N}\,, \tag{B.8}$$

where $u_\imath^\alpha$, with $\imath = 1, \ldots, N$, represent N mutually orthogonal unit vectors in $\mathbb{C}^{N+1}$ so $\bar{z}_\alpha$ is a hyperplane in $\mathbb{C}^{N+1}$: the set of all hyperplanes is the Grassmanian $\mathbb{G}\mathrm{r}_{N;N+1} \cong \mathbb{G}\mathrm{r}_{1;N+1} = \mathbb{CP}^N$ and provides an equivalent construction of $\mathbb{CP}^N$ [58] and it is essentially the fuzzy version of the latter that has been provided in section 4. The $u_\imath^\alpha$ transform as a anti-fundamental representation of $\mathrm{SU}(N)$ on the index $\imath$ and a fundamental of $\mathrm{SU}(N+1)$ on the index α while $\bar{u}_\alpha^\imath$ transform as the corresponding conjugate representations.

We would like to write the left vector fields in (B.4) solely in terms of z^α. To this end, observe that

$$-\left(\bar{u}_\beta^\imath \frac{\partial}{\partial \bar{u}_\alpha^\imath} - \frac{\delta^\alpha{}_\beta}{N+1} \bar{u}_\delta^\imath \frac{\partial}{\partial \bar{u}_\delta^\imath}\right) z^\gamma = \left(z^\alpha \frac{\partial}{\partial z^\beta} - \frac{\delta^\alpha{}_\beta}{N+1} z^\delta \frac{\partial}{\partial z^\delta}\right) z^\gamma \tag{B.9}$$

and the derivatives annihilate $u_\imath^\alpha$ when $\imath = 1, \ldots, N$. So we decompose I into 0 and $\imath = 1, \ldots, N$ and write the left vector fields in equation (B.4) as

$$\begin{aligned}
\mathcal{L}^\alpha{}_\beta &= \left(\bar{u}_\beta^\imath \frac{\partial}{\partial \bar{u}_\alpha^\imath} - \frac{\delta^\alpha{}_\beta}{N+1} \bar{u}_\gamma^\imath \frac{\partial}{\partial \bar{u}_\gamma^\imath}\right) - \left(u_\imath^\alpha \frac{\partial}{\partial u_\imath^\beta} - \frac{\delta^\alpha{}_\beta}{N+1} u_\imath^\gamma \frac{\partial}{\partial u_\imath^\gamma}\right) \\
&= -\left(z^\alpha \frac{\partial}{\partial z^\beta} - \frac{\delta^\alpha{}_\beta}{N+1} z^\delta \frac{\partial}{\partial z^\delta}\right) + \left(\bar{z}_\beta \frac{\partial}{\partial \bar{z}_\alpha} - \frac{\delta^\alpha{}_\beta}{N+1} \bar{z}_\delta \frac{\partial}{\partial \bar{z}_\delta}\right) .
\end{aligned} \tag{B.10}$$

This last form clearly projects trivially to $\mathbb{CP}^N$ and, as for $\mathbb{CP}^1$, decomposes into mutually commuting holomorphic and anti-holomorphic parts,

$$\mathcal{L}^\alpha{}_\beta = L^\alpha{}_\beta - \overline{L^\beta{}_\alpha}\,, \tag{B.11}$$

where

$$L^\alpha{}_\beta = -z^\alpha \frac{\partial}{\partial z^\beta} + \frac{\delta^\alpha{}_\beta}{N+1} z^\delta \frac{\partial}{\partial z^\delta}. \tag{B.12}$$

The Laplacian is

$$\begin{aligned}
-\nabla^2 &= \frac{1}{2} \mathcal{L}^\alpha{}_\beta \mathcal{L}^\beta{}_\alpha \qquad\qquad \text{(B.13)} \\
&= -\partial.\bar{\partial} + \frac{1}{N+1}(z.\partial)(\bar{z}.\bar{\partial}) + \frac{N}{2(N+1)} \left\{(z.\partial)(z.\partial + N + 1) + (\bar{z}.\bar{\partial})(\bar{z}.\bar{\partial} + N + 1)\right\}.
\end{aligned}$$

For the left invariant generators of G that are not in H, we have

$$\mathcal{R}_{\bar{\imath}} := \mathcal{R}^\imath{}_0 = z^\alpha \frac{\partial}{\partial u_\imath^\alpha}, \qquad \mathcal{R}_\imath := \mathcal{R}^0{}_\imath = u_\imath^\alpha \frac{\partial}{\partial z^\alpha}, \tag{B.14}$$

so the N^2 generators of $H = \mathrm{U}(N)$ are

$$[\mathcal{R}_\imath, \mathcal{R}_{\bar{\jmath}}] = u_\imath^\alpha \frac{\partial}{\partial u_\jmath^\alpha} - \delta^\jmath{}_\imath z^\alpha \frac{\partial}{\partial z^\alpha} = \mathcal{R}^\jmath{}_\imath - \frac{\delta^\jmath{}_\imath}{N+1}\left(N z^\alpha \frac{\partial}{\partial z^\alpha} - u_k^\alpha \frac{\partial}{\partial u_k^\alpha}\right). \tag{B.15}$$

The commutator $[\mathcal{R}_\imath, \mathcal{R}_{\bar{\jmath}}]$ generates the holonomy group of tangent space rotations, and so vanishes on functions $f(\bar{z}, z)$ on $\mathbb{CP}^N$ as can be seen by noting that (B.7) gives

$$u_\imath^\alpha \frac{\partial}{\partial u_\jmath^\alpha} \bar{z}_\beta = \delta_\imath{}^\jmath \bar{z}_\beta = \delta_\imath{}^\jmath \bar{z}_\delta \frac{\partial}{\partial \bar{z}_\delta} \bar{z}_\beta \tag{B.16}$$

so

$$[\mathcal{R}_\imath, \mathcal{R}_{\bar{\jmath}}] f(\bar{z}, z) = \delta^\imath{}_\jmath \left(z^\alpha \frac{\partial}{\partial z^\alpha} - \bar{z}_\alpha \frac{\partial}{\partial \bar{z}_\alpha} \right) f(\bar{z}, z) = 0 \tag{B.17}$$

as required.

One can write $\mathcal{R}_\imath$ and $\mathcal{R}_{\bar{\imath}}$ differently noticing that, when acting on elements of the matrix U,

$$\frac{\partial}{\partial u_0^\alpha} = \frac{\partial}{\partial z^\alpha} = \frac{\epsilon_{\alpha\beta_1\cdots\beta_N}}{N!} \frac{\epsilon^{\imath_1\cdots\imath_N}}{N!} \frac{\partial}{\partial \bar{u}^{\imath_1}_{\beta_1}} \cdots \frac{\partial}{\partial \bar{u}^{\imath_N}_{\beta_N}}, \tag{B.18}$$

since this gives $\frac{\partial}{\partial z^\alpha} z^\beta = \delta_\alpha{}^\beta$ and $\frac{\partial}{\partial z^\alpha} u_\imath^\beta = 0$. Thus

$$\begin{aligned} \mathcal{R}_\imath &= \epsilon^{\alpha\beta_1\cdots\beta_N} \bar{u}^1_{\beta_1} \cdots \bar{u}^N_{\beta_N} \frac{\partial}{\partial u_\imath^\alpha} \\ \mathcal{R}_{\bar{\imath}} &= \frac{1}{N!} \epsilon_{\alpha\beta_1\cdots\beta_N} u_\imath^\alpha \frac{\partial}{\partial \bar{u}^1_{\beta_1}} \cdots \frac{\partial}{\partial \bar{u}^N_{\beta_N}} \end{aligned} \tag{B.19}$$

though, in contrast to the case of $\mathbb{CP}^1$ in (A.10), these forms are only valid for expressions that are linear in z^α; a problem that will be considered in detail and rectified in the Fock space construction of appendix C.

C. Recovering the Heisenberg Algebra

The annihilation and creation operators $a^\imath_\alpha$ and $(a^\imath_\alpha)^\dagger = (a^\dagger)^\alpha_\imath$ satisfy

$$[a^\imath_\alpha, (a^\dagger)^\beta_\jmath] = \delta^\imath_\jmath \delta^\beta_\alpha, \tag{C.1}$$

with $\imath = 1, \ldots, N$ an SU(N) index and $\alpha = 1, \ldots, N+1$ an SU($N+1$) index. We defined composite operators as singlets of $su(N)$ by

$$(\widetilde{A}^\alpha)^\dagger = \widetilde{A}^\dagger_\alpha := \frac{1}{N!} \epsilon^{\imath_1\cdots\imath_N} \epsilon_{\alpha\theta_1\cdots\theta_N} (a^\dagger)^{\theta_1}_{\imath_1} \cdots (a^\dagger)^{\theta_N}_{\imath_N}\,. \tag{C.2}$$

These operators enjoy the following commutation relations:

$$[a^\jmath_\beta, \widetilde{A}^\dagger_\alpha] = \frac{1}{(N-1)!} \epsilon_{\alpha\beta\theta_2\cdots\theta_N} \epsilon^{\jmath\imath_1\cdots\imath_{N-1}} (a^\dagger)^{\theta_2}_{\imath_1} \cdots (a^\dagger)^{\theta_N}_{\imath_{N-1}}, \tag{C.3}$$

with

$$\left[\hat{\mathcal{N}}, \widetilde{A}^\dagger_\alpha\right] = \widetilde{A}^\dagger_\alpha\,, \tag{C.4}$$

where

$$\hat{\mathcal{N}} = \frac{1}{N} (a^\dagger)^\alpha_\imath a^\imath_\alpha \tag{C.5}$$

is the reduced number operator and $\widetilde{A}_\alpha$ has unit charge for the U(1) associated with this generator. Furthermore

$$(a^\dagger)^\gamma_\imath [a^\jmath_\gamma, \widetilde{A}^\dagger_\alpha] = \delta^\jmath_\imath \widetilde{A}^\dagger_\alpha\,, \quad \text{i.e.} \quad \left[\hat{J}_\imath{}^\jmath - \delta^\jmath_\imath \hat{\mathcal{N}}, \widetilde{A}^\dagger_\alpha\right] = 0 \tag{C.6}$$

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reflects the fact that $\widetilde{A}_\alpha$ is an $su(N)$ singlet and

$$(a^\dagger)^\gamma_\jmath [a^\jmath_\beta, \widetilde{A}^\dagger_\alpha] = \delta^\gamma_\beta \widetilde{A}^\dagger_\alpha - \delta^\gamma_\alpha \widetilde{A}^\dagger_\beta \quad \text{or} \quad \left[\hat{J}^\gamma{}_\beta, \widetilde{A}^\dagger_\alpha\right] = \delta^\gamma_\beta \widetilde{A}^\dagger_\alpha - \delta^\gamma_\alpha \widetilde{A}^\dagger_\beta \tag{C.7}$$

demonstrates that $\widetilde{A}^\dagger_\alpha$ transforms as the fundamental of $u(N+1)$. Also defining the $su(N)$ singlet states $|\,\widetilde{\boldsymbol{\alpha}}\rangle$ and $|\,\widetilde{\boldsymbol{\alpha}}_k\rangle$ as in (4.12) we therefore have

$$\begin{aligned} \hat{J}_\imath{}^\jmath\,|\,\widetilde{\boldsymbol{\alpha}}\rangle &= \delta^\jmath_\imath \hat{\mathcal{N}}\,|\,\widetilde{\boldsymbol{\alpha}}\rangle \qquad \text{and} \\ \hat{J}^\gamma{}_\beta\,|\,\widetilde{\boldsymbol{\alpha}}\rangle &= L\delta^\gamma_\beta\,|\,\widetilde{\boldsymbol{\alpha}}\rangle - \sum_{k=1}^{L} \widetilde{A}^\dagger_\beta \delta^\gamma_{\alpha_k}\,|\,\widetilde{\boldsymbol{\alpha}}_k\rangle\ . \end{aligned} \tag{C.8}$$

Let us now consider the algebra of $\widetilde{A}^\alpha$ and $\widetilde{A}^\dagger_\beta$. Define

$$D^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}(p) := D^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p} := \sum_{\imath_1\cdots\imath_p \text{distinct}}^{N} (a^\dagger)^{\nu_1}_{\imath_1} a^{\imath_1}_{\mu_1} \cdots (a^\dagger)^{\nu_p}_{\imath_p} a^{\imath_p}_{\mu_p}\ .$$

First observe the lower and upper extreme cases:

$$D^\nu_\mu(1) = (a^\dagger)^\nu_\imath a^\imath_\mu = \hat{J}^\nu{}_\mu\,,$$

are the generators of $\mathrm{U}(N+1)$ and

$$\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}} D^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}(N) = N!\widetilde{A}^\dagger_\beta \widetilde{A}^\alpha\,,$$

where $\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}$ is the $p+1$-delta symbol, the p-delta symbol being defined via

$$\epsilon^{\alpha_1\cdots\alpha_p\gamma_{p+1}\cdots\gamma_{N+1}} \epsilon_{\beta_1\cdots\beta_p\gamma_{p+1}\cdots\gamma_{N+1}} = (N+1-p)!\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p} \tag{C.9}$$

and explicitly

$$\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p} = \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\ldots\delta^{\alpha_p}_{\beta_p} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\ldots\delta^{\alpha_p}_{\beta_p} + \ldots \tag{C.10}$$

with $p!$ terms.

Now separating $\hat{J}^{\nu_{p+1}}{}_{\mu_{p+1}} = (a^\dagger)^{\nu_{p+1}}_\imath a^\imath_{\nu_{p+1}}$ as

$$\hat{J}^{\nu_{p+1}}{}_{\mu_{p+1}} = (a^\dagger)^{\nu_{p+1}}_{\imath_1} a^{\imath_1}_{\mu_{p+1}} + \cdots (a^\dagger)^{\nu_{p+1}}_{\imath_p} a^{\imath_p}_{\mu_{p+1}} + \sum_{\imath_{p+1}\notin\{\imath_1,\ldots,\imath_p\}} (a^\dagger)^{\nu_{p+1}}_{\imath_{p+1}} a^{\imath_{p+1}}_{\mu_{p+1}} \tag{C.11}$$

and substituting into

$$\delta^{\alpha\mu_1\cdots\mu_p\mu_{p+1}}_{\beta\nu_1\cdots\nu_p\nu_{p+1}} D^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p} \hat{J}^{\nu_{p+1}}{}_{\mu_{p+1}} \tag{C.12}$$

we see that due to antisymmetry of the p-delta symbol we have

$$\delta^{\alpha\mu_1\cdots\mu_p\mu_{p+1}}_{\beta\nu_1\cdots\nu_p\nu_{p+1}} (a^\dagger)^{\nu_1}_{\imath_1} a^{\imath_1}_{\mu_1} \cdots (a^\dagger)^{\nu_p}_{\imath_p} a^{\imath_p}_{\mu_j} \left((a^\dagger)^{\nu_{p+1}}_{\imath_k} a^{\imath_k}_{\mu_{p+1}} + \delta^{\nu_{p+1}}_{\mu_{p+1}}\right) = 0 \tag{C.13}$$

for $k = 1, 2, \ldots, p$ and so we obtain the recursion relation

$$\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}} D^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}(p+1) = \delta^{\alpha\boldsymbol{\mu}\mu_{p+1}}_{\beta\boldsymbol{\nu}\nu_{p+1}} D^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}(p) \left(p\delta^{\nu_{p+1}}_{\mu_{p+1}} + \hat{J}^{\nu_{p+1}}{}_{\mu_{p+1}}\right)$$

which gives

$$\widetilde{A}^{\dagger}_{\beta}\widetilde{A}^{\alpha} = \frac{1}{N!}\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\prod_{p=1}^{N}\left((p-1)\delta^{\nu_p}_{\mu_p} + \hat{J}^{\nu_p}{}_{\mu_p}\right) \;. \tag{C.14}$$

Similarly defining $\underline{D}^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$ by interchanging the roles of creation and annihilation operators in $D^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}$, one obtains the recursion relation

$$\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\underline{D}^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}(p+1) = \delta^{\alpha\boldsymbol{\mu}\mu_{p+1}}_{\beta\boldsymbol{\nu}\nu_{p+1}}\underline{D}^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}(p)\Big((N-p)\delta^{\nu_{p+1}}_{\mu_{p+1}} + \hat{J}^{\nu_{p+1}}{}_{\mu_{p+1}}\Big) \;. \tag{C.15}$$

With the initial condition

$$\underline{D}^{\nu}_{\mu}(1) = \hat{J}^{\nu}{}_{\mu} + N\delta^{\nu}_{\mu}\,, \tag{C.16}$$

iterating (C.15) yields

$$\widetilde{A}^{\alpha}\widetilde{A}^{\dagger}_{\beta} = \frac{1}{N!}\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\prod_{p=1}^{N}\left(p\delta^{\nu_p}_{\mu_p} + \hat{J}^{\nu_p}{}_{\mu_p}\right) \;. \tag{C.17}$$

Thus we have that $\widetilde{A}^{\dagger}_{\alpha}\widetilde{A}^{\alpha}$ and $\widetilde{A}^{\alpha}\widetilde{A}^{\dagger}_{\alpha}$ can be expressed as polynomials in the Casimirs of $u(N+1)$ up to C_N.

Furthermore, the commutator is

$$[\widetilde{A}^{\alpha},\,\widetilde{A}^{\dagger}_{\beta}] = \frac{1}{(N-1)!}\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\prod_{p=1}^{N-1}\left(p\delta^{\nu_p}_{\mu_p} + \hat{J}^{\nu_p}{}_{\mu_p}\right)\delta^{\nu_N}_{\mu_N}\,,$$

and the commutator is a polynomial in the Casimirs, C_i, of $u(N+1)$ up to C_{N-1}.

We can use this result to determine how $\widetilde{A}^{\alpha}$ behaves on the reduced Fock space $|\boldsymbol{\gamma}\rangle$. We contract all $\widetilde{A}^{\dagger}_{\alpha}$ with an arbitrary vector x^{α} to obtain $X = x^{\alpha}\widetilde{A}^{\dagger}_{\alpha}$. Then,

$$\widetilde{A}^{\alpha}\,|\,X^{L}\,\rangle = \frac{x^{\beta}}{N!}\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\prod_{p=1}^{N}\left(p\delta^{\nu_p}_{\mu_p} + \hat{J}^{\nu_p}{}_{\mu_p}\right)|\,X^{L-1}\rangle \;. \tag{C.18}$$

Next, we use (C.7) to see that

$$[\hat{J}^{\nu}{}_{\mu} - \delta^{\nu}{}_{\mu}\hat{\mathcal{N}}, X] = -x^{\nu}\widetilde{A}^{\dagger}_{\mu}\,, \tag{C.19}$$

which allows us to substitute for $\hat{J}^{\nu}{}_{\mu}$ in (C.18); then observing that the contribution from the right hand side of (C.19) gives zero, due to antisymmetrization of x^{μ} with x^{β}, we have that $\hat{J}^{\nu}{}_{\mu}$ is replaced with $\hat{\mathcal{N}}\delta^{\nu}_{\mu}$ when acting on $|\,X^{L-1}\rangle$ in (C.18) so that

$$\widetilde{A}^{\alpha}\,|\,X^{L}\,\rangle = \frac{x^{\beta}}{N!}\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\prod_{p=1}^{N}\left((p+\hat{\mathcal{N}})\delta^{\nu_p}_{\mu_p}\right)|\,X^{L-1}\rangle = x^{\alpha}\frac{(\hat{\mathcal{N}}+N)!}{\hat{\mathcal{N}}!}\,|\,X^{L-1}\rangle \;. \tag{C.20}$$

Taking L derivatives with respect to x^{γ} we find

$$\widetilde{A}^{\alpha}\,|\,\boldsymbol{\gamma}\rangle = \frac{(L+N-1)!}{L!}\sum_{i=1}^{L}\delta^{\alpha}{}_{\gamma_i}\,|\,\hat{\boldsymbol{\gamma}}_i\rangle\,. \tag{C.21}$$

Now defining

$$A^{\alpha} := \widetilde{A}^{\alpha}\sqrt{\frac{\hat{\mathcal{N}}!}{(\hat{\mathcal{N}}+N-1)!}} \tag{C.22}$$

we have, when acting on states $|\boldsymbol{\gamma}\rangle$ of the reduced Fock space,

$$A^{\dagger}_{\alpha}A^{\alpha}\,|\boldsymbol{\gamma}\rangle = \hat{\mathcal{N}}\,|\boldsymbol{\gamma}\rangle \tag{C.23}$$

and

$$[A^{\alpha}, A^{\dagger}_{\beta}]\,|\boldsymbol{\gamma}\rangle = \delta^{\alpha}_{\beta}\,|\boldsymbol{\gamma}\rangle\ . \tag{C.24}$$

Hence we find that A^{α} and $A^{\dagger}_{\beta}$ obey the Heisenberg Algebra.

From the above we can write

$$A^{\dagger}_{\beta}A^{\alpha} = \hat{\mathcal{N}}\frac{1}{N!}\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\prod_{p=0}^{N-1}\left(\frac{p+\hat{J}}{p+\hat{\mathcal{N}}}\right)^{\nu_p}_{\mu_p} \tag{C.25}$$

and

$$A^{\alpha}A^{\dagger}_{\beta} = (\hat{\mathcal{N}}+1)\frac{1}{N!}\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\prod_{p=1}^{N}\left(\frac{p+\hat{J}}{p+\hat{\mathcal{N}}}\right)^{\nu_p}_{\mu_p} \tag{C.26}$$

so that

$$[A^{\alpha}, A^{\dagger}_{\beta}] = \frac{1}{N!}\delta^{\alpha\boldsymbol{\mu}}_{\beta\boldsymbol{\nu}}\prod_{p=1}^{N-1}\left(\frac{p+\hat{J}}{p+\hat{\mathcal{N}}}\right)^{\nu_p}_{\mu_p}\left(\frac{N(\hat{\mathcal{N}}+1)+(1-N)\hat{J}}{N+\hat{\mathcal{N}}}\right)^{\nu_N}_{\mu_N}\ . \tag{C.27}$$

Also observe that (C.8) becomes

$$(\hat{J}^{\gamma}{}_{\beta} + \widetilde{A}^{\dagger}_{\beta}\widetilde{A}^{\gamma})\,|\widetilde{\boldsymbol{\alpha}}\rangle = \hat{\mathcal{N}}\delta^{\gamma}_{\beta}\,|\widetilde{\boldsymbol{\alpha}}\rangle\ . \tag{C.28}$$

References

[1] A. Connes, *Noncommutative geometry*, Academic Press (1994).

[2] J.M. Gracia-Bondia, J.C. Varilly and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser (2001).

[3] J. Madore, *An introduction to noncommutative differential geometry and physical applications*, L.M.S. Lect. Note Ser. **257** (2000) 1.

[4] G. Landi, *An introduction to noncommutative spaces and their geometry*, Lecture Notes in Physics, Monographs mp51, Springer-Verlag (1997) [hep-th/9701078].

[5] A.P. Balachandran, G. Mangano, A. Pinzul and S. Vaidya, *Spin and statistics on the Groenwald-Moyal plane: Pauli-forbidden levels and transitions*, *Int. J. Mod. Phys.* **A 21** (2006) 3111 [hep-th/0508002].

[6] S. Doplicher, K. Fredenhagen and J.E. Roberts, *The quantum structure of space-time at the Planck scale and quantum fields*, *Commun. Math. Phys.* **172** (1995) 187 [hep-th/0303037].

[7] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[8] P. Aschieri, J. Madore, P. Manousselis and G. Zoupanos, *Dimensional reduction over fuzzy coset spaces*, *JHEP* **04** (2004) 034 [hep-th/0310072].

[9] B.P. Dolan and C. Nash, *Chiral fermions and* Spin(C) *structures on matrix approximations to manifolds*, *JHEP* **07** (2002) 057 [hep-th/0207007].

[10] B.P. Dolan and C. Nash, *The standard model fermion spectrum from complex projective spaces*, *JHEP* **10** (2002) 041 [hep-th/0207078].

[11] A.P. Balachandran, S. Kurkcuoglu and S. Vaidya, *Lectures on fuzzy and fuzzy SUSY physics*, hep-th/0511114.

[12] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

[13] R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162].

[14] W. Taylor, *M(atrix) theory: matrix quantum mechanics as a fundamental theory*, *Rev. Mod. Phys.* **73** (2001) 419 [hep-th/0101126].

[15] D. Karabali and V.P. Nair, *Quantum hall effect in higher dimensions, matrix models and fuzzy geometry*, *J. Phys.* **A 39** (2006) 1273 [hep-th/0606161].

[16] A.P. Balachandran, T.R. Govindarajan and B. Ydri, *The fermion doubling problem and noncommutative geometry*, *Mod. Phys. Lett.* **A 15** (2000) 1279 [hep-th/9911087].

[17] X. Martin, *A matrix phase for the* ϕ^4 *scalar field on the fuzzy sphere*, *JHEP* **04** (2004) 077 [hep-th/0402230].

[18] F. Garcia Flores, D. O'Connor and X. Martin, *Simulating the scalar field on the fuzzy sphere*, PoS(LAT2005)262 [hep-lat/0601012].

[19] J. Medina, W. Bietenholz, F. Hofheinz and D. O'Connor, *Field theory simulations on a fuzzy sphere: an alternative to the lattice*, PoS(LAT2005)263 [hep-lat/0509162].

[20] M. Panero, *Quantum field theory in a non-commutative space: theoretical predictions and numerical results on the fuzzy sphere*, *SIGMA* **2** (2006) 081 [hep-th/0609205].

[21] M. Panero, *Numerical simulations of a non-commutative theory: the scalar model on the fuzzy sphere*, hep-th/0608202.

[22] H. Grosse, M. Maceda, J. Madore and H. Steinacker, *Fuzzy instantons*, *Int. J. Mod. Phys.* **A 17** (2002) 2095 [hep-th/0107068].

[23] H. Steinacker, *Quantized gauge theory on the fuzzy sphere as random matrix model*, *Nucl. Phys.* **B 679** (2004) 66 [hep-th/0307075].

[24] P. Castro-Villarreal, R. Delgadillo-Blando and B. Ydri, *A gauge-invariant UV-IR mixing and the corresponding phase transition for* U(1) *fields on the fuzzy sphere*, *Nucl. Phys.* **B 704** (2005) 111 [hep-th/0405201].

[25] H. Grosse and H. Steinacker, *Finite gauge theory on fuzzy* $\mathbb{CP}^2$, *Nucl. Phys.* **B 707** (2005) 145 [hep-th/0407089].

[26] D. O'Connor and B. Ydri, *Monte Carlo simulation of a NC gauge theory on the fuzzy sphere*, *JHEP* **11** (2006) 016 [hep-lat/0606013].

JHEP07(2007)007

[27] W. Bietenholz, J. Nishimura, Y. Susaki and J. Volkholz, *A non-perturbative study of* 4*D* U(1) *non-commutative gauge theory: the fate of one-loop instability*, *JHEP* **10** (2006) 042 [hep-th/0608072].

[28] F.A. Berezin, *General concept of quantization*, *Commun. Math. Phys.* **40** (1975) 153.

[29] J. Hoppe, *Quantum theory of a massless relativistic surface and a two dimensional bound state problem*, Ph.D. thesis (1982).

[30] J. Madore, *The fuzzy sphere*, *Class. and Quant. Grav.* **9** (1992) 69.

[31] J. Arnlind, M. Bordemann, L. Hofer, J. Hoppe and H. Shimada, *Fuzzy Riemann surfaces*, hep-th/0602290.

[32] A.P. Balachandran, B.P. Dolan, J.-H. Lee, X. Martin and D. O'Connor, *Fuzzy complex projective spaces and their star-products*, *J. Geom. Phys.* **43** (2002) 184 [hep-th/0107099].

[33] B.P. Dolan and O. Jahn, *Fuzzy complex grassmannian spaces and their star products*, *Int. J. Mod. Phys.* **A 18** (2003) 1935 [hep-th/0111020].

[34] B.P. Dolan, D. O'Connor and P. Prešnajder, *Fuzzy complex quadrics and spheres*, *JHEP* **02** (2004) 055 [hep-th/0312190].

[35] S. Ramgoolam, *On spherical harmonics for fuzzy spheres in diverse dimensions*, *Nucl. Phys.* **B 610** (2001) 461 [hep-th/0105006].

[36] B.P. Dolan and D. O'Connor, *A fuzzy three sphere and fuzzy tori*, *JHEP* **10** (2003) 060 [hep-th/0306231].

[37] H. Grosse, C. Klimčík and P. Prešnajder, *Topologically nontrivial field configurations in noncommutative geometry*, *Commun. Math. Phys.* **178** (1996) 507 [hep-th/9510083].

[38] P. Prešnajder, *The origin of chiral anomaly and the noncommutative geometry*, *J. Math. Phys.* **41** (2000) 2789 [hep-th/9912050].

[39] A.M. Perelomov, *Generalized coherent states and their applications*, Springer-Verlag (1986).

[40] F.A. Berezin, *Quantization in complex symmetric spaces*, Izvestija **9** (1975) 341.

[41] B.P. Dolan, D. O'Connor and P. Prešnajder, *Matrix* ϕ^4 *models on the fuzzy sphere and their continuum limits*, *JHEP* **03** (2002) 013 [hep-th/0109084].

[42] S. Kurkcuoglu and C. Saemann, *Drinfeld twist and general relativity with fuzzy spaces*, *Class. and Quant. Grav.* **24** (2007) 291 [hep-th/0606197].

[43] S. Baez, A.P. Balachandran, B. Ydri and S. Vaidya, *Monopoles and solitons in fuzzy physics*, *Commun. Math. Phys.* **208** (2000) 787 [hep-th/9811169].

[44] A.P. Balachandran and S. Vaidya, *Instantons and chiral anomaly in fuzzy physics*, *Int. J. Mod. Phys.* **A 16** (2001) 17 [hep-th/9910129].

[45] H. Grosse, C.W. Rupp and A. Strohmaier, *Fuzzy line bundles, the chern character and topological charges over the fuzzy sphere*, *J. Geom. Phys.* **42** (2002) 54 [math-ph/0105033].

[46] U. Carow-Watamura, H. Steinacker and S. Watamura, *Monopole bundles over fuzzy complex projective spaces*, *J. Geom. Phys.* **54** (2005) 373 [hep-th/0404130].

[47] E. Hawkins, *Quantization of equivariant vector bundles*, *Commun. Math. Phys.* **202** (1999) 517 [q-alg/9708030].

[48] E. Hawkins, *Geometric quantization of vector bundles*, *Commun. Math. Phys.* **215** (2000) 409 [math.QA/9808116].

[49] H. Grosse and A. Strohmaier, *Noncommutative geometry and the regularization problem of 4D quantum field theory*, *Lett. Math. Phys.* **48** (1999) 163 [hep-th/9902138].

[50] *Encyclopedic Dictionary of Mathematics*, Second Edition, by the Mathematical Society of Japan, edited by Kiyosi Itô, MIT Press (1987).

[51] S. Chaturvedi and N. Mukunda, *The Schwinger* SU(3) *construction - I: multiplicity problem and relation to induced representations*, *J. Math. Phys.* **43** (2002) 5262 [quant-ph/0204119].

[52] S. Chaturvedi and N. Mukunda, *The Schwinger* SU(3) *construction - II: relations between Heisenberg-Weyl and* SU(3) *coherent states*, *J. Math. Phys.* **43** (2002) 5278 [quant-ph/0204120].

[53] S. Murray and C. Saemann, *Quantization of flag manifolds and their supersymmetric extensions*, hep-th/0611328.

[54] S.-C. Zhang and J.-p. Hu, *A four dimensional generalization of the quantum hall effect*, *Science* **294** (2001) 823 [cond-mat/0110572].

[55] J.-p. Hu and S.-C. Zhang, *Collective excitations at the boundary of a 4D quantum Hall droplet*, *Phys. Rev.* **B 66** (2002) 125301 [cond-mat/0112432].

[56] D. Karabali and V.P. Nair, *Quantum Hall effect in higher dimensions*, *Nucl. Phys.* **B 641** (2002) 533 [hep-th/0203264].

[57] B.P. Dolan, I. Huet, S. Murray and D. O'Connor, in preparation.

[58] P.S. Howe and G.G. Hartwell, *A superspace survey*, *Class. and Quant. Grav.* **12** (1995) 1823.